Dynamic Identification in VARs

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January 2023
Preliminary

Abstract

Most macroeconomic models, view economic outcomes as being generated by a combination of endogenous and exogenous dynamic forces. In particular, the exogenous forces are generally modeled as a set of independent dynamics processes. In this paper we begin by showing that this dual dynamic structure is sufficient to identify the entire set of structural impulse responses inherent to any such model. No extra restrictions are needed. We then use this result to suggest how it can be used to evaluate common SVAR restrictions (impact restrictions, long run restrictions and proxy-VAR).

Keyword: Structural Shocks, Dynamic Identification, SVARs, DSGE models.

JEL Code: C32, E32

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1 Introduction

Macroeconomists are often interested in knowing how the economy reacts to different types of shocks (see Ramey (2016) and Stock and Watson (2017) for very detailed accounts of the recent macroeconomic literature). There are two main approaches to look at this issue. On the one hand, one can build a fully specified Dynamic Stochastic General Equilibrium (DSGE) model, estimate it using full information methods (see *e.g.* Smets and Wouters (2007), Christiano et al. (2010) and Lindé et al. (2016)), and look at its implied impulse response functions (IRF). These IRF are generally referred to as structural impulse responses. The advantage of such an approach is that the identification of shocks is generally granted (see *e.g.* Canova and Sala (2009), Iskrev (2010), Komunjer and Ng (2011)), given the many restrictions imposed by the (generally small scale) structural model. One caveat though —the flip side of the same coin— is that a DSGE model imposes many restrictions on the data and, consequently, is prone to mis-specification. On the other hand, one can follow the Structural Vector Autoregressive (SVAR) literature (see Section 4 in Stock and Watson (2016) and Kilian and Lütkepohl (2016) for a complete review) and impose a more limited set of identifying restrictions —restrictions more loosely motivated by theory or alternatively motivated by institutions— to derive structural impulse responses using a VAR.[1] SVARs are less prone to mis-specification, but mapping their implications into the language of models and exogenous structural shocks is not uncontroversial.

It is important to note that the SVAR approach is aimed at obtaining the same objects than those obtained using a DSGE, that is, it is aimed at recovering structural impulse responses that can be interpreted as being the outcome of an economy subjected to particular exogenous driving forces. It is this last observation that we want to exploit in this paper. In particular, we will show that when a VAR is viewed as the reduced form of a DSGE model, then one can immediately obtain the full set of structural impulses responses without the need of any additional identifying restrictions. Because of this property, most identifying restrictions used in the SVAR literature can be visually evaluated or formally tested. For instance, this allows for the evaluation of the variety of identifying restrictions imposed in SVARs (Short-run, Long-run, proxy-variables ...).

Dynamic Identification. Identification is best understood from the simple recognition that IRF implied by DSGE models reflect both propagation mechanisms associated with the functioning of the economy as well as external dynamics associated with the exogenous driving forces. For these exogenous driving forces to have a clear structural interpretation, it is usually assumed that the exogenous processes are linearly independent, both contemporaneously and over time. The fact that the processes for the exogenous driving forces are restricted is key. For example, a common assumption is that the exogenous driving forces are governed by linearly independent AR(1) processes. As we shall show, because DSGE models share this structure, one can recover structural IRF directly from the implied VAR without the need of any of the

[1] We use the generic term VAR, which, in our case, also includes Vector Error Correction Models (VECM).

additional assumption on the loading of the shocks used in the SVAR literature. In other words, the specification of the lag structure and the set of variables of a SVAR is sufficient to identify a set of structural shocks. No additional assumptions are needed to obtain the set of structural IRF. Because it explicitly makes use of restrictions on the dynamic structure of the underlying model, we dub that particular SVAR a "D-SVAR". By *dynamic structure*, we mostly mean the process of the exogenous forcing variables in the model, although the state variables and the lag structure also (obviously) need to be specified. By *identification of structural shocks*, we mean that there generically exists a unique[2] vector of mutually orthogonal shocks in the D-SVAR that satisfies the restrictions imposed by the dynamic structure of the DSGE model. In loose and over-simplified terms, if the economy is moved by exogenous variables that follow linearly independent AR(1) processes, then the economy follows a D-SVAR and identification of structural shocks is granted. This theoretical result will be shown in Section 3.

Testing commonly used SVAR restrictions. It is worth mentioning that the structural shocks identified by our D-SVAR approach remain unlabelled (nothing in the identification assigns a name (technology, fiscal, etc…) to a structural shock). While standard SVAR restrictions (impact, long-run, sign …) can be used to label them *ex-post*, our approach can also be used to test these latter restrictions.

To fix ideas, consider the bi-variate environment examined in the seminal paper by Blanchard and Quah (1989). This paper aimed at deriving the impulse responses associated with supply and demand shocks. The identification restriction used to separate the two shocks under consideration imposed that a demand shock has no long-run (permanent) effect on GDP, while a supply shock may. Instead, our approach allows us to first obtain the two unique structural impulse response consistent with a DSGE, and then to examine the extent to which the Blanchard and Quah (1989)'s restrictions are consistent with our D-SVAR.

We will provide three examples drawn from the literature to show how our approach can be used to evaluate SVAR strategies. We first present the Blanchard and Quah (1989) example discussed above. Then we examine the proxy VAR strategy used in Gertler and Karadi (2015) to identify monetary shocks by exploiting a high frequency instrument (see Kuttner (2001), Gürkaynak et al. (2005), Bernanke and Kuttner (2005) and Gürkaynak et al. (2007) for early work on High Frequency Identification of monetary policy shocks). Finally, we assess the validity of the impact restrictions used in Christiano et al. (1999, 2005) to identify monetary policy shocks. We will show that, for two of these examples (Blanchard and Quah (1989) and Christiano et al. (1999)), the identifying restrictions cannot be rejected within our D-SVAR, while they are in the proxy-VAR of Gertler and Karadi (2015).

Related Literature. Our identification result relates to several papers including, among others, McGrattan (2010), Pagan and Robinson (2019), Bai and Wang (2015) and Gourieroux and Jasiak (2022). While several of our theoretical results have precedents in the literature, our

[2]Uniqueness is up to the sign and/or a permutation of the shocks.

contribution is to establish how and when the implicit assumptions put on the data generating process by SVARs allow for the identification of the full set of structural impulse responses.

McGrattan (2010) derives conditions for identification of an unrestricted state-space representation associated with a specific small-scale Real Business Cycle model.[3] In particular, the paper shows that, when the model includes a permanent technology shock and a stationary labor wedge shock (in the form of a labor income tax), the unrestricted state-space representation is identified. Our paper provides conditions for identification in a broader class of DSGE models, admitting a VAR or a VARMA representation of the solution. Pagan and Robinson (2019) note that SVARs may face difficulty to properly uncover the loading matrix of DSGE models because standard estimation of SVAR models avoids imposing the type of statistical restrictions commonly used in DSGE models —*e.g.* that structural shocks follow mutually orthogonal univariate autoregressive processes. The two authors discuss conditions for local identification in SVARs when the autoregressive matrix is diagonal and shocks are normalized. Our formal analysis shows more generically the conditions on the autoregressive matrix that allows to identify the structural shocks. Bai and Wang (2015) study identification in dynamic factor models similar to our unrestricted state space representation. Their approach, in line with the conventional way of identifying shocks in the VAR literature, imposes restrictions on the loading matrix while leaving unrestricted the autoregressive matrix of factors. In this paper, we take the opposite viewpoint and determine which type of organisation of the autoregressive matrix allows to freely identify the loading matrix in the state-space representation. Finally, Gourieroux and Jasiak (2022) provide conditions for identification in multivariate undetermined convoluted systems when the exogenous shocks (the "*sources*" in their terminology) follow linearly independent autoregressive processes of order one and when there is no intrinsic dynamics of the endogenous variables. They show that when the autoregressive parameters are distinct, the loading matrix (the *"mixing matrix"* in their terminology) is identified. Our paper departs from theirs in at least three dimensions. First, we consider a larger class of dynamic models and makes connections with the DSGE literature. Second, we extend the identification problem to non diagonal autoregressive processes. Third, we determine conditions for partial identification when the practitioner seeks to identify only a subset of structural shocks.

Outline. The paper is structured as follows. Section 2 presents the main results of the paper. It shows how the D-SVAR representation can be derived, explains heuristically why it is identified and presents an application. Section 3 formally proves local identification. Section 4 discusses estimation and inference in the D-SVAR setup. Section 5 illustrates the use of D-SVARs to assess SVARs in the context of monetary policy shocks. A last section concludes. All proofs are reported in an appendix.

[3] See also Kascha and Mertens (2009) for simulation experiments in a similar setup.

2 A Primer on D-SVARs

This section shows how to derive our D-SVAR representation, while explaining its relationship with a (linearised) DSGE model. It then explains intuitively why this D-SVAR should be identified, by checking the necessary order condition, leaving proof of identification to the next section.[4] Finally, it presents a simple application to an output growth–unemployment VAR.

2.1 A Basic Setup

Let us assume that the Data Generating Process (DGP hereafter) is an economic model (typically a DSGE model) of the type

$$\begin{aligned} X_t &= M_1 X_{t-1} + M_2 \mathbb{E}_t[X_{t+1}] + M_3 Z_t, \\ Z_t &= R Z_{t-1} + \varepsilon_t. \end{aligned} \tag{1}$$

where $\mathbb{E}_t[\cdot]$ denotes the expectation operator conditional on period t information set, X_t is a $n_x \times 1$ vector of endogenous variables and Z_t is a $n_z \times 1$ vector of structural shocks. Those shocks are assumed to be autoregressive of order one.[5] The structural innovations ε_t are normally distributed, with zero mean and their covariance matrix is identity. Note that this implies that the loading matrix M_3 encapsulates the size of the shocks. The vector X_t splits between the $(n_y \times 1)$ vector Y_t of observed variables and the $(n_k \times 1)$ vector of unobserved (latent) variables K_t. Note that some substitutions might be needed to obtain a system featuring as many observed variables as shocks ($n_y = n_z$).

Matrices M_1, M_2, M_3 are functions of the vector of deep parameters, θ, and encapsulate any cross-equation restrictions imposed by the micro-foundations of the DSGE model. Note in particular that those matrices M_i may contain some zero elements. Finally matrix R gathers all the parameters pertaining to the dynamics of the shock processes. In this section, it is assumed that all the variables in X_t are observable, while the shocks Z_t are not. The case of non-observable state variables will be dealt with in Section .7 of the Online Appendix. The solution of the model admits[6] the following state space representation

$$\begin{aligned} K_t &= G K_{t-1} + F Z_t, \\ Y_t &= \Pi_{yk} K_{t-1} + \Pi_{yz} Z_t, \\ Z_t &= R Z_{t-1} + \varepsilon_t. \end{aligned} \tag{2}$$

where G, F, Π_{yk} and Π_{yz} are functions of M_1, M_2, M_3 and R, and therefore of θ and R, as represented by the mapping

$$(G, F, \Pi_{yk}, \Pi_{yz}, R) = \Phi(\theta, R).$$

[4] Here we refer to the first order condition for identification. However, local identification may still be possible using higher order conditions (see Sargan (1983) and Dovonon and Hall (2018)).

[5] To keep exposition simple at this stage, we present only the case of a model with one lead and one lag and an order one process for shocks. Conceptually, everything extends to higher order models.

[6] This implicitly assumes that the dynamic system admits a saddle path. When the system is locally indeterminate, the Z_t vector can be extended to capture extrinsic uncertainty.

Shall System (2) be identified, one can then go one step further and identify the model parameters, provided that the mapping Φ is invertible.

Having set the stage for system (2), we are now in a position to discuss the identification of shocks.

2.2 Heuristic Approach to Identification

We consider the case where the state vector X_t only consists of observed variables Y_t, which, as we will show, can be directly written as a VAR. The case of latent endogenous state variables is treated in full generality in the next section. We also assume that there are as many such observed variables as shocks ($n = n_y = n_z$). In this case, $\Pi_{yk} = I$ and $\Pi_{yz} = 0$, such that the system reduces to

$$\begin{aligned} X_t &= GX_{t-1} + FZ_t, \\ Z_t &= RZ_{t-1} + \varepsilon_t. \end{aligned} \tag{3}$$

Eliminating Z_t, System (3) can be written as a SVAR(2) process, that we dub a D-SVAR:

$$X_t = \left(G + FRF^{-1}\right) X_{t-1} - FRF^{-1}GX_{t-2} + F\varepsilon_t. \tag{4}$$

Estimating a VAR(2) on the data, one can obtain the non structural VAR representation:

$$X_t = \Gamma_1 X_{t-1} + \Gamma_2 X_{t-2} + \nu_t. \tag{5}$$

where ν_t is a vector of canonical innovations with covariance matrix Σ_ν. The representation (5) is referred to as the non structural VAR. Matrices Γ_1, Γ_2 and Σ_ν are functions of G,F and R according to the mapping

$$(\Gamma_1, \Gamma_2, \Sigma_\nu) = \Psi(G, F, R).$$

Note that provided the D-SVAR representation can be recovered –*i.e.* if the mapping Ψ is invertible, one can compute the theoretical impulse responses functions of the structural model, the variance decomposition, conditional correlations ...

Identifying the D-SVAR (4) means recovering matrices G, F and R from Γ_1, Γ_2 and Σ_ν. Absent any restrictions, each matrix contains n^2 elements, so that $3n^2$ unknown coefficients need to be recovered. The available information in the non-structural VAR is given by $(\Gamma_1, \Gamma_2, \Sigma_\nu)$. The system of equations that determines the elements of F,G and R is, using (4) and (5):

$$\left\{ \begin{array}{lcl} \Gamma_1 &=& G + FRF^{-1}, \\ \Gamma_2 &=& -FRF^{-1}G, \\ \Sigma_\nu &=& FF'. \end{array} \right. \tag{6}$$

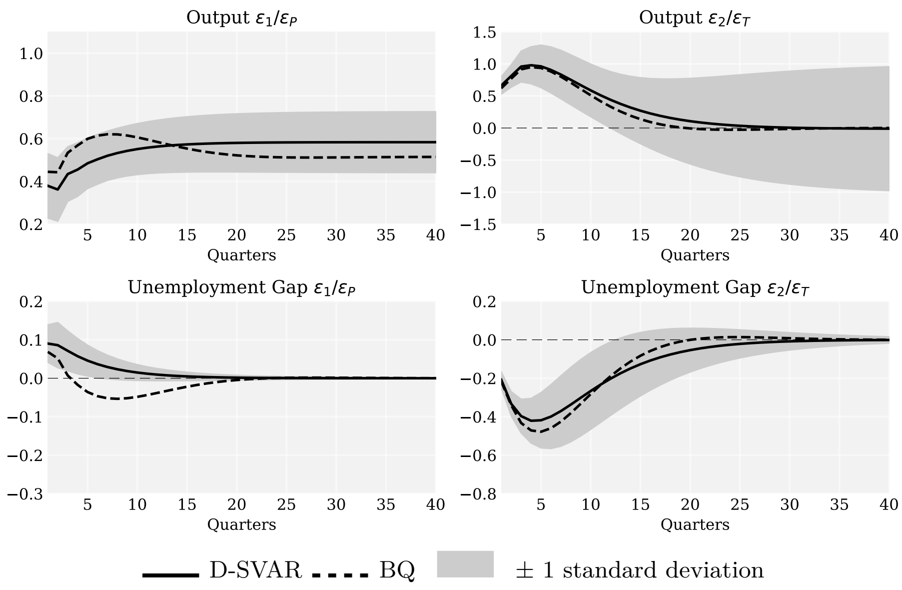
Because Σ_ν is symmetric, this system only provides with $3n^2 - \frac{n(n-1)}{2}$ independent equations for $3n^2$ unknowns. This is the well-known problem of the identification of shocks in SVARs. If one adds some extra identifying restrictions (at least $\frac{n(n-1)}{2}$), then F, G and R can be identified. Of course, this order condition is only necessary, and a rank condition also needs to be satisfied (see Section 3). For now, let us simply count the number of restrictions and check a necessary condition for identification. A restriction typically imposed in the VAR literature assumes that F is lower triangular, which amounts to restrict the effect of shocks on impact (see Sims (1980)). This puts exactly $\frac{n(n-1)}{2}$ restrictions, so that the VAR is just identified. But, shall the loading matrix F be obtained from solving a standard DSGE, F is a complicated function of the matrices M_1, M_2, M_3 and R, and may not necessarily comply with the lower triangular assumption unless some specific assumptions are placed on the timing of agent's decisions (see e.g. Christiano et al. (2005)). Likewise, restricting the long-run, as in Blanchard and Quah (1989), imposes a particular structure on the loading matrix F that not all DSGE share.

Our D-SVAR approach does not hinge on restricting the loading matrix F but rather relies on assumptions placed on the dynamic structure of the shocks only —*i.e.,* on the autoregressive matrix R. To fix ideas, let us assume that the shocks in Z_t are mutually orthogonal at all leads and lags – *i.e.* that R is a diagonal matrix with distinct diagonal elements. In other words, let us assume that all the shocks in the DSGE model follow linearly independent AR(1) processes. While this assumption may appear very restrictive at first sight, it is shared by a vast majority of the DSGE literature. In that case, the necessary order condition is satisfied: R only consists of n non zero diagonal elements, so that we have $2n^2+n$ unknowns to determined. Note however that, as we will show in the next section, counting restrictions does not guarantee identification. For example, a diagonal autoregressive matrix with identical elements (*i.e.* $R = \rho I_n$) cannot be identified as, in that case, F drops from the first two equations of System (6). We will follow another strategy to prove formally identification in the next section

2.3 Application to a Bivariate VAR

Here we apply our dynamic identification to a bivariate VAR featuring the growth rate of output per capita, Δy_t, and a measure of the unemployment rate gap, u_t, computed as the gap between the actual and non-cyclical rate of unemployment. Blanchard and Quah (1989) (BQ hereafter) used a similar VAR to uncover the permanent, ε_P, and transitory, ε_T, component of output by imposing that the latter has no long-run effect on the level of output. We estimate a VAR for the 1960Q1–2007Q4 period using two lags of data, as selected by BIC, and recover the D-SVAR representation. As we will show in Section 4, we use by an Asymptotic Least Squares estimation method (see Corollary 1) using the unrestricted VAR as auxiliary model. A J-test can be designed, that does not reject the over–identifying restrictions imposed by the D-SVAR (see Section 4 for details). Impulse responses to our two structural shocks are displayed in Figure 1. Table 1 reports the associated forecast error variance decomposition at various horizons.

We uncover an interesting and familiar pattern. There is a shock, ε_2, that increases output and decreases unemployment on impact. Then the response of output is hump-shaped and goes back to almost zero in the long-run. This shock explains more than 80% of unemployment volatility at any horizon, about 75% of the volatility of output on impact, but about 0% in the long-run. The other shock, ε_1, exerts a permanent effect on output and little effect in unemployment. The two shocks look pretty much like the permanent and temporary shocks of BQ. This is confirmed by the dash lines on Figure 1, which correspond to the BQ identification.



Sample is 1960Q1-2007Q4. y is the real GDP, u is the unemployment rate gap. Estimation is done with (Δy,u) using two lags. The grey area represents 68% confidence bands obtained from 1,000 Bootstrap replications.

Figure 1: Comparing the D-SVAR with Blanchard and Quah (1989) Identification

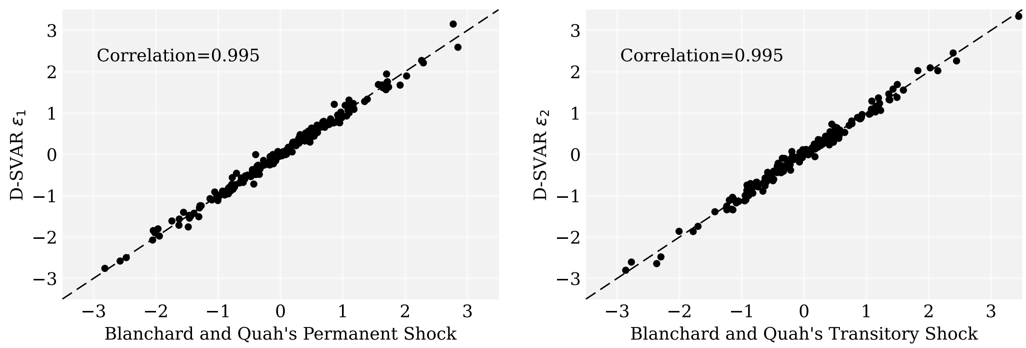
Table 1: Forecast Error Variance Decomposition, $(\Delta y, u)$, D-SVAR and Blanchard and Quah (1989)

	Output				Unemployment gap			
Horizon	ε_1	ε_2	ε_P	ε_T	ε_1	ε_2	ε_P	ε_T
1	24.7	75.3	34.0	66.0	15.9	84.1	9.2	90.8
4	18.4	81.6	27.0	73.0	4.7	95.3	1.3	98.7
8	22.3	77.7	31.3	68.7	2.6	97.4	1.2	98.8
20	42.1	57.9	49.1	50.9	2.0	98.0	1.7	98.3
∞	99.8	0.2	100.0	0.0	2.0	98.0	1.7	98.3

Sample is 1960Q1-2007Q4. Estimation is done with (Δy,u) using two lags, where y is the real GDP and u is the unemployment rate gap. ε_1 and ε_2 correspond to the D-SVAR, ε_P and ε_T to Blanchard and Quah (1989).

Responses are indeed very similar. Figure 2 shows scatter plots of the BQ's permanent and transitory shocks against ε_1 and ε_2: the correlation between the D-SVAR and BQ shocks is almost perfect in both cases.

As we will make it explicit in Section 4, the estimation of the D-SVAR allows to test the BQ identification restriction. If one restricts the data to be generated by a model in which the two latent shocks are linearly independent AR(1) processes, then, as can be seen on Figure 1, one cannot reject at 0% that one shock has a permanent effect on output. Figure 1 seems to indicate that the impact response of the unemployment gap differs across the two identifications. However, a formal test of the null hypothesis of equality between the two IRF does not reject the null hypothesis, with a p-value of 40%.



Sample is 1960Q1-2007Q4. Estimation is done with $(\Delta y, u)$ using two lags, where y is the real GDP and u is the unemployment rate gap.

Figure 2: Correlation between D-SVAR and BQ shocks

Our dynamic identification hence recovers dynamics that are extremely similar to Blanchard and Quah (1989), whose identifying restrictions can be tested (and not rejected) under the D-SVAR.[7]

3 D-SVAR Identification

This section formally proves local identification of the D-SVAR. We start by fixing some notations that will be used throughout the proof, we then analyze identification relying on covariance matrix restrictions only and finally add dynamic restrictions.

[7] As illustrated in the Online Appendix .10, an advantage of our approach though is that the identification of the shocks does not require the estimation of the spectral density of, at least, one variable at frequency 0 —an object which is usually hard to estimate and at best very imprecise (see, *e.g.* Fernald (2007)).

3.1 Setup

Consider an economy whose DGP is described by the following state-space representation (we abstract from constant vectors without any loss of generality)

$$\underbrace{K_t}_{n_k\times 1} = \underbrace{G}_{n_k\times n_k}\underbrace{K_{t-1}}_{n_k\times 1} + \underbrace{F}_{n_k\times n_z}\underbrace{Z_t}_{n_z\times 1}, \tag{7}$$

$$\underbrace{Y_t}_{n_y\times 1)} = \underbrace{\Pi_{yk}}_{n_y\times n_k}\underbrace{K_t}_{n_k\times 1} + \underbrace{\Pi_{yz}}_{n_y\times n_z}\underbrace{Z_t}_{n_z\times 1}, \tag{8}$$

$$\underbrace{Z_t}_{n_z\times 1} = \underbrace{R}_{n_z\times n_z}\underbrace{Z_{t-1}}_{n_z\times 1} + \underbrace{\varepsilon_t}_{n_z\times 1}, \tag{9}$$

where the $(n_y \times 1)$ vector Y_t gathers all observed variables, $(n_k \times 1)$ vector K_t collects all of possibly unobserved (latent) state variables, Z_t represents the $(n_z \times 1)$ vector of unobserved exogenous variables and ε_t is the $(n_z \times 1)$ vector of structural innovations to Z_t. In particular, ε_t satisfies $\mathbb{E}_{t-1}\varepsilon_t = 0$, where $\mathbb{E}_{t-1}$ denotes the expectation operator conditional on the information set of histories until period $t-1$, *i.e.* all past realizations and histories of $\{K_t, Y_t, Z_t\}$.[8] This framework is general enough to represent the (log-)linear solution of most (dynamic general) equilibrium model, including among others DSGE models. This solution usually depends on a limited number of parameters, that we denote by θ gathering all "*deep*" structural parameters (representing preferences, technology, institutions, policies ...), together with the stochastic process of the exogenous forcing variables present in the structural model. In this case, the matrices in the state-space representation (7)–(9) will be functions of θ. In this paper, we do not consider the identification of θ, but instead the identification of the state-space parameters that freely enter in the matrices of the state-space representation. We denote this vector of state-space parameters ψ. It must be clear to the reader that system (7)–(9) imposes less restrictions than the (possibly underlying) DSGE model. The only restrictions that we will explore apply to the matrix R and the covariance matrix of the structural innovations ε_t in Equation (9). Finally, the shocks ε_t are zero mean weak white noise processes with covariance matrix $\mathbb{E}(\varepsilon_t\varepsilon_t') = \Sigma_\varepsilon$.

System (7)-(9) can be rewritten in the more compact form

$$S_{t+1} = AS_t + B\varepsilon_{t+1}, \tag{10}$$

$$Y_t = \Pi S_t, \tag{11}$$

with

$$\underbrace{S_t}_{n_s\times 1} = \begin{bmatrix} K_t \\ Z_t \end{bmatrix}, \qquad \underbrace{A}_{n_s\times n_s} = \begin{bmatrix} G & FR \\ 0_{n_z\times n_k} & R \end{bmatrix}, \qquad \underbrace{B}_{n_s\times n_z} = \begin{bmatrix} F \\ I_{n_z} \end{bmatrix} \quad \text{and} \quad \underbrace{\Pi}_{n_y\times n_s} = \begin{bmatrix} \Pi_{yk} & \Pi_{yz} \end{bmatrix},$$

where $n_s = n_k + n_z$. System (10)-(11) can be expressed as the Fernández-Villaverde et al. (2007) ABCD representation

$$S_{t+1} = AS_t + B\varepsilon_{t+1}, \tag{12}$$

$$Y_{t+1} = CS_t + D\varepsilon_{t+1}, \tag{13}$$

[8] Shall some elements K_t be observed, those elements should be reassigned to vector Y_t and the matrices Π_{yk} and Π_{yz} should be adjusted accordingly.

where $C = \Pi A$ and $D = \Pi B$. Note that, in the sequel, we consider the case where the number of observables is equal to the number of shocks. Some assumptions need to be placed on the ABCD representation (12)-(13).

Assumption 1 *For any $z \in \mathbb{C}$, $det(I - Az) = 0$ implies $|z| > 1$.*

Assumption 1 restricts the class of matrices A to those with eigenvalues lying inside the unit circle. Under Assumption 1 and using (10)-(11) and/or (12)-(13), the process $\{Y_t\}$ admits the following infinite Vector Moving Average (VMA) representation:

$$Y_t = \Pi(I - AL)^{-1}B\varepsilon_t = \left[C(I - AL)^{-1}BL + D\right]\varepsilon_t = H(L, \psi)\varepsilon_t$$

where $H(z, \psi) = \sum_{j=0}^{\infty} h(j; \psi)z^j$ is the transfer function and L denotes the lag operator. For every $\psi \in \Psi$, $\mathbb{E}(Y_t) = 0$ and

$$\mathbb{E}(Y_t Y_s') \equiv \Gamma(s - t; \psi) = \sum_{j=0}^{\infty} h(j; \psi)\Sigma_\varepsilon h(j + s - t; \psi)',$$

for all $t, s \geq 1$, where $\psi = (vec(A)', vec(B)', vec(C)', vec(D)', vech(\Sigma_\varepsilon)')'$ is the vector collecting all the parameters of the state-space representation (7)–(9).

For any weakly stationary process $\{Y_t\}$ implied by Assumption 1 and under the assumption that the shocks ϵ_t are Gaussian, the unconditional mean and auto-covariance function completely characterize the properties of the process. Let us therefore define the auto-covariance generating function as:

$$\Omega(z, \psi) = \sum_{j=-\infty}^{\infty} \Gamma(j; \psi)z^j,$$

for any $z \in \mathbb{C}$. Evaluating $\Omega(z, \psi)$ at $z = \exp(i\omega)$ for any $\omega \in [-\pi, \pi]$ and rescaling it by $(2\pi)^{-1}$ yields the spectral density of the observable $\{Y_t\}$ which is always positive semi-definite. To simplify, we hereafter refer to $\Omega(z, \psi)$ as the spectral density as well.

As will be clear later, it will prove useful to defined observational equivalence for this class of multivariate covariance stationary process. We closely follow Komunjer and Ng (2011) and define it with respect to the entire auto-covariance function of the observable (or the spectral density).

Definition 1 *Two sets of state-space parameters ψ and $\widetilde{\psi}$ are observationally equivalent if $\Omega(z; \psi) = \Omega(z; \widetilde{\psi})$, for all $z \in \mathbb{C}$ or, equivalently, $\Gamma(j; \psi) = \Gamma(j; \tilde{\psi})$ at all $j \geq 0$.*

In other words, two stationary state-space models are observationally equivalent is they share the same auto-covariance (spectral) properties. This then allows us to define local identification.

Definition 2 *The state-space representation (10)-(11) is locally identifiable from the spectral density of Y_t (or equivalently from the auto-covariances of Y_t) at $\psi \in \Psi$ if there exists an open neighborhood of ψ such that for every $\tilde{\psi}$ in this neighbourhood, ψ and $\widetilde{\psi}$ are observationally equivalent if and only if $\tilde{\psi} = \psi$.*

In state space system, the spectral density function can be simply obtained from the transfer function and the covariance matrix of the shocks, Σ_ε, as

$$\Omega(z, \psi) = H(z;\psi)\Sigma_\varepsilon H(z^{-1};\psi)',$$

where, in our ABCD representation of the dynamics,

$$H(z;\psi) = \Pi(I - Az)^{-1}B \equiv C(I - Az)^{-1}B + D,$$

with $z = \exp(i\omega)$ for any $\omega \in [-\pi, \pi]$. As explained in Komunjer and Ng (2011), this representation of the spectral density makes it clear that equivalent spectral density function can obtain because *(i)* for given Σ_ε, two distinct vectors of state space parameters, ψ and $\widetilde{\psi}$, yield the same transfer function ($H(z;\psi) = H(z;\tilde{\psi})$) or *(ii)* many pairs of $H(z;\psi)$ and Σ_ε give rise to the same spectral density.

In general, the state-space parameter ψ is not identifiable from the second order moments of the observable variables. As an illustration, let us consider the following two state-space representations (K_t is observable):

$$\mathcal{S} = \begin{cases} K_t &= GK_{t-1} + FZ_{t-1}, \\ Y_t &= \Pi_{yk}K_t + \Pi_{yz}Z_t, \\ Z_t &= RZ_{t-1} + \varepsilon_t, \end{cases} \qquad \widetilde{\mathcal{S}} = \begin{cases} K_t &= GK_{t-1} + \widetilde{F}\widetilde{Z}_{t-1}, \\ Y_t &= \Pi_{yk}K_t + \widetilde{\Pi}_{yz}Z_t, \\ \widetilde{Z}_t &= \widetilde{R}\widetilde{Z}_{t-1} + \widetilde{\varepsilon}_t, \end{cases}$$

where $\widetilde{Z}_t = U^{-1}Z_t$, $\widetilde{F} = FU$, $\widetilde{\Pi}_{yz} = \Pi_{yz}U$, $\widetilde{R} = U^{-1}RU$ and $\widetilde{\Sigma}_\varepsilon = U^{-1}\Sigma_\varepsilon U^{-1\prime}$ for some full rank matrix U. The two representations $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ are observationally equivalent with respect to the spectral function since $\Omega(z,\psi) = \Omega(z,\widetilde{\psi})$ for all $z \in \mathbb{C}$ where the vectors ψ and $\tilde{\psi}$ gather, respectively, the elements of the vectorization of matrices defining, respectively, $\mathcal{S}$ and $\widetilde{\mathcal{S}}$.

Following Komunjer and Ng (2011) (see Proposition 1-S), the following property obtains in the case of the $ABCD$ representation

Property: *Two distinct vectors of state-space parameters $\psi, \widetilde{\psi} \in \Psi$ are observationally equivalent respective to the transfer function and the spectral density if and only if there exists a full rank $n_s \times n_s$ matrix T and a full rank $n_z \times n_z$ matrix U such that $\tilde{A} = TAT^{-1}$, $\widetilde{B} = TBU$, $\widetilde{C} = CT^{-1}$, $\widetilde{D} = DU$ and $\Sigma_{\widetilde{\varepsilon}} = U^{-1}\Sigma_\varepsilon U^{-1\prime}$.*

This property obtains as follows. First, the equalities $\tilde{A} = TAT^{-1}$, $\widetilde{B} = TB$, $\widetilde{C} = CT^{-1}$ are necessary and sufficient for the equivalence of the transfer function $H(z;\tilde{\psi}) = H(z;\psi)$. Sufficiency follows directly from the observation of the transfer function in the ABCD representation

$$\begin{aligned} H(z;\tilde{\psi}) &= \widetilde{C}(I - \tilde{A}z)^{-1}\widetilde{B} + D \\ &= CT^{-1}(I - TAzT^{-1})^{-1}TB + D \\ &= CT^{-1}T(I - Az)^{-1}T^{-1}TB + D = H(z;\psi). \end{aligned}$$

The necessary condition follows directly from a well known result in control theory under the condition of minimality of the state-space representation (See Theorem 3.10 in Antsaklis and Michel (1997) and Chapter 8 in Gouriéroux and Monfort (1995)). A system is minimal if and only if it is controllable and observable (see Appendix .1 for formal definitions), implying that, among all systems leading to the same output spectral density, it is driven by the minimal number of state variables. This equivalence class is function of a non-singular transformation that corresponds to a rotation of the state, i. e., $T^{-1}S_t$.

The equivalence of the spectral density is obtained for a full rank matrix U such that:

$$H(z;\psi)\Sigma_\varepsilon H(z^{-1};\psi)' = H(z;\psi)UU^{-1}\Sigma_\varepsilon U^{-1\prime}U'H(z^{-1};\psi)'.$$

Fixing $\widetilde{B} = BU$, $\widetilde{D} = DU$ and $\Sigma_{\tilde{\varepsilon}} = U^{-1}\Sigma_\varepsilon U^{-1\prime}$, this yields $H(z;\tilde{\psi}) = H(z;\psi)U = DU + C\left[I - Az\right]^{-1} BU$. Observational equivalence follows immediately.

We therefore established that identification of ψ or a subset of ψ cannot obtain without placing additional restrictions on the covariance matrix, Σ_ε. This is what we do in the next section.

3.2 Covariance Matrix Restrictions

This section provides a key proposition on local identification when restrictions are placed on the covariance matrix only. In particular, we consider the case where the structural innovations are mutually orthogonal and their covariance matrix is normalised to the identity matrix such that $\mathbb{E}(\varepsilon_t\varepsilon_t') = I_{n_z}$ – a common identifying assumption in the SVAR literature (see Ramey (2016)). In this case, matrices F and Π_{yz} in (7)–(9) encapsulate any scale effect from the shocks —*i.e.* contains information about the volatility of the shocks. A direct implication of this assumption is that the only admissible matrix U which allows for $\Sigma_{\widetilde{\varepsilon}} = U^{-1}\Sigma_\varepsilon U^{-1\prime} = I_{n_z}$ is an orthonormal matrix—*i.e.* $UU' = I_{n_z}$ (see Corollary 1 of Kocięcki and Kolasa (2018)).

We further make the following assumption, that adapts Assumption 1 to our initial state space representation (7)–(9).

Assumption 1′ *For any $z \in \mathbb{C}$, $det(I - Az) = 0$ implies $|z| > 1$ and matrices G and R have no eigenvalues in common.*

Since matrix A is block triangular (see System (10)–(11)), we have $det(I - Az) = det(I - Gz)det(I - Rz)$. Matrices G and R have all their eigenvalues lying within the unit circle and do not have any common eigenvalue. The latter assumption is necessary to disentangle the dynamics of the latent variables K_t from those of the exogenous process Z_t. Endowed with Assumption 1′, the next proposition shows that the rotation matrix T is block upper triangular.

Proposition 1 *Under Assumption 1′, we have*

- *When the state variables K_t are unobserved, the full rank matrix T has the form*

$$T = \begin{bmatrix} T_{11} & T_{12} \\ 0_{n_z \times n_k} & V \end{bmatrix}, \tag{14}$$

  with T_{11} a full rank $(n_k \times n_k)$ matrix.

- *When the state variables K_t are observed, the full rank matrix T has necessarily the following form:*

$$T = \begin{bmatrix} I_{n_k} & 0_{n_k \times n_z} \\ 0_{n_z \times n_k} & V \end{bmatrix}, \tag{15}$$

In both cases, V is an orthonormal $(n_z \times n_z)$ matrix such that $VV' = I_{n_z}$ and $V = U' = U^{-1}$ defined above.

Proof : See Appendix .2.

Proposition 1 implies that, as long as state variable K_t is observed, matrices G and Π_{yk} can be identified using the observed spectral density function.[9] However, it does not allow for proper identification of the loading matrix F relying on the properties of Z_t. Indeed, a direct implication of the proposition is that there exists at least one equivalent exogenous process to (9):

$$Z_t = VRV'Z_{t-1} + \varepsilon_t. \tag{16}$$

Observational equivalence in terms of transfer function (and spectral density) holds if and only if sub-matrix V in matrix T defined in (15) is orthonormal ($VV' = I_{n_z}$). In that case, pre-multiplying (16) by V' we get

$$\widetilde{Z}_t = R\widetilde{Z}_{t-1} + \widetilde{\varepsilon}_t,$$

where $\widetilde{Z}_t = V'Z_t$, $\widetilde{\varepsilon}_t = V'\varepsilon_t$ and $\mathbb{E}(\widetilde{\varepsilon}_t\widetilde{\varepsilon}_t') = I_{n_z}$. In other words, the sole knowledge of the spectral properties of Z_t is not sufficient to identify F. Further (dynamic) restrictions need to be placed on the autoregressive matrix R.

3.3 Covariance Matrix and Dynamic Restrictions

This section focuses on the local identification of the exogenous process Z_t relying on second order moments information (spectral density of observables Y_t). Our strategy is to show that the only admissible permutation matrix in equation (16) is $V = I_{n_z}$ (up to changes of sign and/or permutation of the identity matrix) and hence that there indeed exists only one Z_t process that is compatible with the spectral properties of observables Y_t. We examine the following four cases (commonly encountered in the DSGE literature):

1. R is a diagonal matrix;

2. R is a lower (identically upper) triangular matrix;

3. R is a symmetric matrix;

4. R is a block diagonal matrix with blocks corresponding to cases 1 and 2;

and prove the identification problem in each case.

[9] The observed spectral density (or equivalently the auto-covariance) function can be obtained by the estimation of a VAR, a VARMA or by the ML estimation of a state space representation.

3.3.1 R is a Diagonal Matrix

The case of a diagonal matrix is of particular interest. It implies, together with the restriction on the covariance matrix, that all processes in the Z_t vector are mutually orthogonal at any leads and lags. While this assumption may sound very restrictive, it actually corresponds to the common practice in the DSGE literature, and hence echoes economic theory. The next proposition derives the sufficient condition for local identification.

Proposition 2 *If R is a diagonal matrix with distinct diagonal elements ($r_{i,i} \neq r_{j,j}, \forall i \neq j$) then the state-space model (10)-(11) is locally identifiable.*

Proof : See Appendix .3.

In other words, the loading matrix F and the autoregressive matrix R are locally identifiable if all the autoregressive parameters of the n_z linearly independent forcing variables are all different. Henceforth, if one has in mind a "standard" DSGE model featuring mutually orthogonal shocks at any leads and lags, dynamic identification easily obtains. From an intuitive point of view, identification obtains because, given an economic structure, differences in the persistence of shocks implies that the impulse response functions to each shock all bring different information regarding the dynamics. To see this more concretely, it may prove useful to consider the following example.

Example: Identification of Demand/Supply Shocks in a New-Keynesian Model. Consider the following textbook 3-equation New Keynesian (NK) model (see Galí (2015)) featuring two structural shocks

$$\begin{aligned} y_t &= \mathbb{E}_t y_{t+1} - (i_t - E_t[\pi_{t+1}]) + z_{1,t}, \\ \pi_t &= \beta \mathbb{E}_t[\pi_{t+1}] + \kappa y_t + z_{2,t}, \\ i_t &= \phi_\pi \pi_t, \end{aligned}$$

where y_t, π_t and i_t denote respectively aggregate output, the rate of inflation and the nominal interest rate and $\mathbb{E}_t[\cdot]$ denotes the conditional expectation operator. Parameter $\beta \in (0, 1)$ is the discount factor, $\kappa \geq 0$ denotes the slope of the Phillips curve and ϕ_π is the degree of aggressiveness of monetary policy to inflation. The random shock $z_{1,t}$ can be interpreted as a demand shock shifting the IS curve, whereas $z_{2,t}$ is a cost-push shock shifting the Phillips curve. For expositional purposes, we assume that the demand shock, $z_{1,t}$ is serially uncorrelated (as a benchmark case), while $z_{2,t}$ exhibits serial correlation. Assuming the Taylor principle holds ($\phi_\pi > 1$), the solution takes the form

$$X_t = F Z_t,$$

where F is a 2×2 matrix that depends on the structural parameters and the persistence of the cost-push shock (ρ). The vector $X_t = (y_t, \pi_t)'$ contains the two endogenous variables and $Z_t = (z_{1,t}, z_{2,t})'$ is the vector of the two structural shocks which is assumed to follow the autoregressive process

$$Z_t = RZ_{t-1} + \varepsilon_t \quad \text{with} \quad R = \begin{bmatrix} 0 & 0 \\ 0 & \rho \end{bmatrix} \quad \text{and} \quad \varepsilon_t = \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \end{bmatrix},$$

where ε_t is a zero mean weak noise and where we impose the normalisation $E(\varepsilon_t \varepsilon_t') = I_2$.

Our problem is then to identify the vector of five parameters $\psi = \{\rho, f_{11}, f_{12}, f_{21}, f_{22}\}$ from the auto-covariance function of y_t and π_t. Each element of the vector X_t can be expressed as a linear combination of the innovations of the shocks as

$$y_t = f_{11}\varepsilon_{1,t} + f_{12}\sum_{i=0}^{\infty} \rho^i \varepsilon_{2,t-i}$$
$$\pi_t = f_{21}\varepsilon_{1,t} + f_{22}\sum_{i=0}^{\infty} \rho^i \varepsilon_{2,t-i}$$

from which the auto-covariances of y_t and π_t can easily be obtained. For instance the variance and auto-covariances of output express as (similarly for inflation)

$$\gamma_y(0) = f_{11}^2 + \frac{f_{12}^2}{1-\rho^2},$$
$$\gamma_y(h) = \rho^h \frac{f_{12}^2}{1-\rho^2} \quad \text{for} \quad h > 0.$$

Note that computing the ratio $\gamma_y(h+1)/\gamma_y(h)$ for any $h > 0$ allows to immediately identify ρ. Given ρ, the knowledge of any $\gamma_y(h)$ for $h > 0$ is sufficient to identify f_{12} (up to its sign). Then, f_{11} (up to its sign) straightforwardly obtains from $\gamma_y(0)$. Using the same approach with the auto-covariance function of inflation identifies f_{21} and f_{22}. It is worth noting that when $\rho = 0$, so the two shocks $z_{1,t}$ and $z_{2,t}$ display the same dynamic properties and the parameters f_{ij} are not identified. Indeed, in this case, the model reduces to

$$\begin{bmatrix} y_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \end{bmatrix},$$

which is not identifiable from the covariance matrix of X_t (3 moments to identify 4 parameters). This example therefore illustrates in what sense the dynamic structure of exogenous forcing variables is key to identify the state–space representation.

3.3.2 R is a Lower Triangular Matrix

We now extend the autoregressive matrix R to the case of lower triangular shape, hence relaxing its diagonal feature. Let us define the matrix

$$\Lambda = \begin{bmatrix} \lambda_1 I_{n_{z1}} & 0 & \cdots & 0 \\ 0 & \lambda_2 I_{n_{z2}} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_L I_{n_{zL}} \end{bmatrix},$$

with $\lambda_i \neq \lambda_j$ for $i \neq j$, and n_{zl} be positive integers for $l = 1, \ldots, L$ such that $n_{z1} + \ldots + n_{zL} = n_z$ and $n_{zl} = m(\lambda_l)$ denotes the multiplicity of eigenvalue λ_l.

Proposition 3 *If R is a lower triangular matrix with the same main diagonal as Λ and all elements on the first sub-diagonal are different from zero, i.e. $r_{i+1,i} \neq 0$ for $i = 1, \ldots, n_z - 1$, the state-space model is locally identifiable.*

Proof : See Appendix .4.

Proposition 3 indicates that, contrary to the strict diagonal case, the lower triangular case does not require all diagonal elements to be different to permit identification. Should there be some identical elements on the diagonal, they should appear in consecutive order —*i.e.* all gathered in one of the matrices $\lambda_\ell I_{n_{z\ell}}$. The proposition also implies that while not all elements below the diagonal have to be non-zero, those lying on the first sub-diagonal have to be non zero. Again, intuitively, the condition for proper local identification is that all shocks should lead to distinguishable dynamics (as captured for example by IRF), hence some elements have to be distinct to guarantee that different shocks generate different dynamics.

3.3.3 R is a Symmetric Matrix

For simplicity, we restrict the presentation the case with two shocks. The matrix R is symmetric with the same value on the diagonal, and the same value on the anti-diagonal. While such a configuration may appear as a curiosity at first sight, it is actually of very practical interest in the international macroeconomic literature since the seminal work of Backus et al. (1992) [10]. For this specification, the autoregressive matrix takes the following form:

$$R = \begin{bmatrix} \rho & \tau \\ \tau & \rho \end{bmatrix} \text{ with } \tau \neq 0. \tag{17}$$

The previous non-zero restriction on τ is critical. When $\tau = 0$, matrix (17) reduces to a diagonal matrix with identical elements and therefore does not satisfy Proposition 2. Provided $\tau \neq 0$, the following proposition holds.

Proposition 4 *For the 2×2 symmetric matrix R (eq. 17), the state-space model is locally identifiable.*

In particular, and contrary to the diagonal case, we show in the Appendix .5 that the diagonal elements must necessarily be identical for local identification of the state-space model.[11]

3.3.4 Partial Identification.

Finally, Proposition 5 establishes that the model is partially identifiable. In other words, even though some shocks may not be identified, it is still possible to identify a subset (Online Appendix .8 provides a simple illustrative example).

[10] This dynamic structure of autoregressive matrix has widely been used, among others, by Backus et al. (1994), Baxter and Crucini (1995), Heathcote and Perri (2002) and Kehoe and Perri (2002).

[11] Appendix .13 offers an illustration of the symmetric autoregressive matrix, R, with an application to the international transmission of shocks between the US and the Euro area.

Proposition 5 *In the case of a block diagonal organisation of the R matrix with one block corresponding to one of the preceding cases, the state–space model is partially locally identifiable for this block.*

Proof : See Appendix .6.

4 Estimation and Inference

This section discusses estimation and inference in the D-SVAR approach and shows how this approach can be used to evaluate DSGE and SVAR models.

4.1 Estimation

To simplify the exposition, and without loss of generality, let us consider the simplified state space representation

$$X_t = FZ_t \tag{18}$$

$$Z_t = RZ_{t-1} + \varepsilon_t \quad , \tag{19}$$

where $\mathbb{E}(\varepsilon_t \varepsilon_t') = I_{nz}$. In the sequel, we will restrict ourselves to cases where the parameter vector $\psi = (vec(F)', vec(R)')'$ is indeed locally identified, such that the conditions for the validity of Propositions 2-5 are satisfied. Vector ψ can thus be estimated either by Maximum Likelihood (ML) from (18)-(19) or equivalently by a two step Asymptotic Least Square (ALS) approach (See Corollary 1 below). Let us denote $\widehat{\psi}_T$ the ML estimator of ψ for a sample size T. Absent any unobserved variables, the D-SVAR representation rewrites as a VAR(1) model:

$$X_t = \left(FRF^{-1}\right) X_{t-1} + F\varepsilon_t \tag{20}$$

where the F is identified using the dynamic structure of unobserved structural shocks. So the loading matrix F is obtained without any restriction. Consider now the reduced-form VAR(1) representation :

$$X_t = \Gamma X_{t-1} + u_t \tag{21}$$

with $E(u_t) = 0$ and $E(u_t u_t') = \Sigma_u$. This implies that $u_t = F\varepsilon_t$.

Our D-SVAR representation imposes as many or more restrictions on the dynamic structure of the data X_t than the unrestricted VAR model. This offers an opportunity to use the information contained in the parameters of the unrestricted estimated VAR model (21) to estimate ψ in the D-SVAR (18)-(19). Let us define $\eta = (vec(\Gamma)', vec(\Sigma_u)')'$ and the binding function $\widetilde{\eta}(\psi)$. We derive a version of the Corollary in Gouriéroux and Monfort (1995) (Chap. 10, Section 10.4.2, Corollary 10.2) adapted to our D-SVAR.

Corollary 1 *Let $\widehat{\eta}_T$ be a consistent and asymptotically normal estimator of η from (21) and let $\eta(\psi)$ be the binding function. The ALS estimator $\widehat{\psi}_T$ obtained by solving*

$$\min_{\psi} \left[\widehat{\eta}_T - \widetilde{\eta}(\psi)\right]' S_T \left[\widehat{\eta}_T - \widetilde{\eta}(\psi)\right] \quad , \tag{22}$$

where S_T is an estimator of the inverse of the asymptotic covariance matrix of $\widehat{\eta}_T$, is a consistent estimator of ψ and is asymptotically equivalent to the ML estimator $\widehat{\psi}_T$ of ψ obtained from (18)-(19).

Intuitively, Corollary 1 states that estimating the parameter vector ψ in one direct step (ML approach) or relying on a two step procedure using η as an auxiliary parameter (ALS approach) yields asymptotic equivalent estimator of ψ. The two-step estimation uses the constraints $\Gamma = (FRF^{-1})$ and $\Sigma_u = FF'$ to uncover the elements of the R and F matrices. This corollary therefore illustrates the strong connection between our approach and standard VAR modelling.

Consider now the DSGE model that underlies system (18) and (19). This DSGE model imposes cross-equation restrictions on the elements of the $F(\theta)$ matrix together with those contained in the $R(\theta)$ matrix. Let us define the binding function $\widetilde{\psi}(\theta)$ that expresses the vector of state-space parameters, ψ, as a function of θ. Under the conditions provided in Komunjer and Ng (2011), vector of parameters θ is also identifiable and can thus be estimated by ML, a usual practice in the applied macroeconomic literature. Let us denote by $\widehat{\theta}_T$ the ML estimator of θ. Because our D-SVAR imposes less restrictions on the state-space representation (18) and (19) than the DSGE model and provided dim $\theta <$ dim ψ, vector ψ can be used as an auxiliary parameter to estimate θ. The following corollary, again adapted from Gouriéroux and Monfort (1995), holds.

Corollary 2 *Let $\widehat{\psi}_T$ be the ML estimator of ψ from the unconstrained state-space version of the representation (18) and (19) and $\widetilde{\psi}(\theta)$ the binding function. The estimator $\widetilde{\theta}_T$ obtained by solving*

$$\min_{\theta} \left[\widehat{\psi}_T - \widetilde{\psi}(\theta)\right]' S_T \left[\widehat{\psi}_T - \widetilde{\psi}(\theta)\right] \quad ,$$

where S_T converges to the inverse of the asymptotic covariance matrix of $\widehat{\psi}_T$ which is given by the information matrix $\mathcal{I}(\psi)$ of the log-likelihood function, is asymptotically equivalent to the ML estimator $\widehat{\theta}_T$ of θ obtained from the constrained state-space version of the representation (18) and (19)

Estimating θ is one step or by a two step procedure using ψ as an auxiliary parameter yields asymptotically equivalent estimator of θ. Our unrestricted state-space representation features less restrictions than the DSGE model and thus contains potentially useful information about the relevance of the structural restrictions imposed by the DSGE model. This corollary illustrates the tight relationship between the DSGE model and the D-SVAR.

4.2 Inference

The D-SVAR offers the possibility to conduct statistical inference both on DSGE and SVAR models. Let us first consider the DSGE and our D-SVAR, and let us remind the reader that the D-SVAR imposes no cross-equation restrictions during estimation. Using ML estimates of the two models, it is then possible to test the relevance of the cross-equation restrictions imposed by the DSGE model, and therefore guide modelling. Let us consider the null hypothesis that the DSGE model can mimic the unconstrained state-space representation (18) and (19), $H_0 \ : \ \psi = \widetilde{\psi}(\theta)$. A Wald-type statistic to test for this null hypothesis is then given by:

$$W_T = T(\hat{\psi}_T - \widetilde{\psi}(\hat{\theta}_T))'\mathcal{I}(\hat{\psi}_T)(\hat{\psi}_T - \widetilde{\psi}(\hat{\theta}_T)) \tag{23}$$

which is asymptotically distributed as χ^2 with $p-q$ degrees of freedom under the null hypothesis, where $p = \dim\, \psi$ and the assumption that $rank\left(\frac{\partial\psi(\theta)}{\partial\theta'}\right) = \dim\, \theta = q$, where $\mathcal{I}(\hat{\psi}_T)$ is an estimator of the information matrix evaluated at the unconstrained estimator $\hat{\psi}_T$ (see Gouriéroux and Monfort (1995), Section 17.4.1).[12] A score test and a Likelihood Ratio test can also be constructed (see Gouriéroux and Monfort (1995), Section 17.4.2 and Section 17.4.3).

As a particular but interesting case, a test for the equality of the loading matrix F (restricted (DSGE) and unrestricted (D-SVAR)) can also be performed. The associated statistic is given by

$$W_T^F = T\left(vec(\hat{F}_T) - vec(\tilde{F}(\hat{\theta}_T)\right)'\mathcal{I}^{11}(\hat{\psi}_T)\left(vec(\hat{F}_T) - vec(\tilde{F}(\hat{\theta}_T))\right)$$

where $vec(F)$ is a p_1-vector. $\mathcal{I}^{11}(\hat{\psi}_T)$ is an estimator of the corresponding block to F of the information matrix and $\tilde{F}(\theta)$ is the binding function linking θ to the loading matrix F. Under the assumption that $rank\left(\frac{\partial vec(\tilde{F}(\theta))}{\partial\theta'}\right) = q_1$ with $q_1 < p_1$ this statistics is asymptotically distributed as a χ^2 with $p_1 - q_1$ degrees of freedom under the null hypothesis. Since, the loading matrix F collects the impact response of each variable to each shock, this test immediately assesses the relevance of the structural restrictions. Inspecting point by point the impact responses and/or the overall dynamic responses is also straightforward.

Let us now consider the D-SVAR and VAR models. First and foremost, a specification test (J-test) can be performed in the case of over-identification, *i.e.* when the dimension of the vector η is greater than the dimension of ψ, by multiplying the objective function (22) by the number of observations. This allows to assess whether the dynamic restrictions imposed by the D-SVAR model are satisfied and hence evaluate the reliability of our D-SVAR approach regarding an unconstrained VAR model.

The D-SVAR also offers the opportunity to assess the relevance of various identification schemes used in SVAR modelling. In particular, it allows to test general null hypotheses on the loading matrix F. For example, one may be interested in the relevance of the timing imposed by short-run restrictions. In this case, the null hypothesis writes $H_0 : Hvec(F) = 0$, where the selection matrix H is such that the elements above the diagonal of F are all equals to zero. Likewise, matrix H can be adapted to test for long-run restrictions à la Blanchard and Quah (1989). A Wald statistic can then be computed with a consistent estimator of the appropriate variance covariance matrix. Likewise, one may be interested in testing for the dynamic response to a particular shock, as identified using two competing identification schemes.

[12] Under possible misspecification, the estimator of the information matrix $\mathcal{I}(\hat{\psi}_T)$ can be replaced by an estimator of the inverse of the sandwich formulae $\mathcal{J}(\hat{\psi}_T)^{-1}\mathcal{I}(\hat{\psi}_T)\mathcal{J}(\hat{\psi}_T)^{-1}$ where $\mathcal{I}(\hat{\psi}_T)$ is an estimator of the variance covariance matrix of the score and $\mathcal{J}(\hat{\psi}_T)$ an estimator of minus the second derivative of the log-likelihood function.

This can be achieved with a Wald statistic and using the appropriate asymptotic distribution or by bootstrapping techniques as proposed by Inoue and Kilian (2016).[13]

5 Testing for SVARs Restrictions: Two Examples With Monetary Policy Shocks

In this section we revisit two seminal papers that both proposed to identify monetary policy shocks through the lens of the D-SVAR. The first one, Gertler and Karadi (2015), relies on an external instrument to identify the shock — the so-called proxy VAR approach. The second, Christiano et al. (1999), identifies a monetary policy shock by imposing zero restrictions on its impact effect on key economic variables.

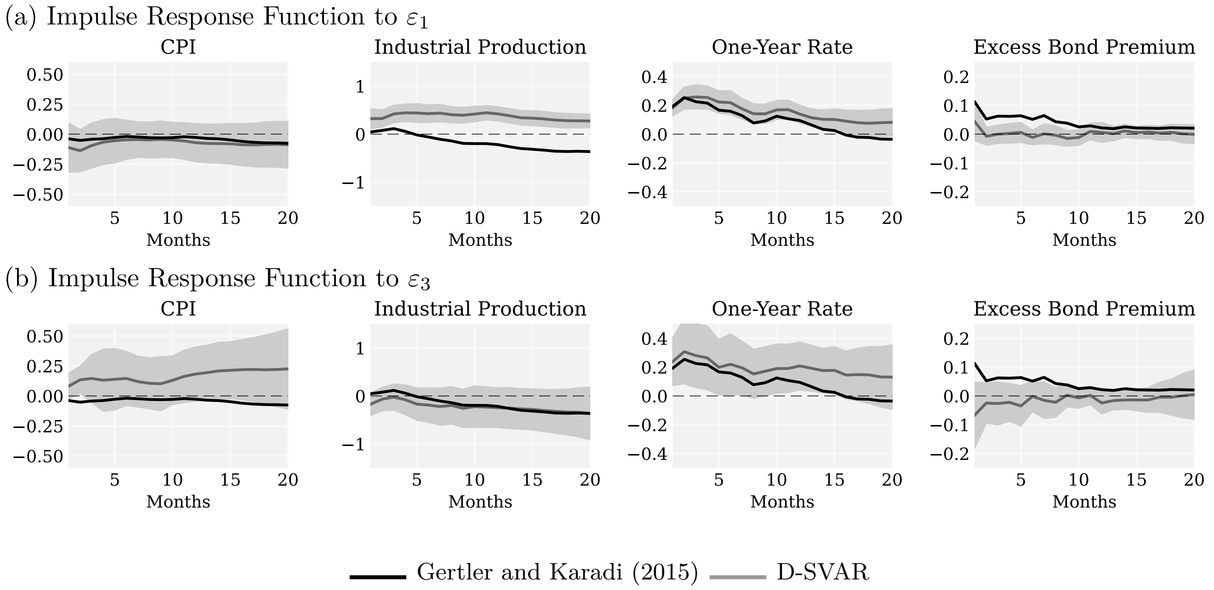
5.1 Revisiting Gertler and Karadi (2015) Proxy VAR

In this section, we revisit Gertler and Karadi (2015), who identified a monetary policy shock relying on a proxy-VAR approach with an external instrument (see Beaudry and Saito (1998), Stock (2008), Stock and Watson (2012) and Mertens and Ravn (2013) among others). Such an approach avoids imposing timing restrictions on both the behavior and the impact of the policy rate. The instrumental variable needs to satisfy two assumptions to identify a given structural shock: *i)* the instrument must be *relevant*, *i.e.* the contemporaneous correlation between the structural shock and the external instrument must be non-zero; *ii)* the instrument must be *exogenous*, *i.e.* the instrument must be uncorrelated with the other structural shocks. According to Gertler and Karadi (2015), an advantage of proxy VARs is that it does not impose a special organization (and thus restrictions) of the loading matrix F. This is also the case in our D-SVAR, and it is therefore interesting to compare the two approaches.

We first replicate Gertler and Karadi (2015) and estimate a VAR featuring the log consumer price index, the log industrial production, the one year government bond rate, and Gilchrist and Zakrajšek (2012) excess bond premium. The data are evaluated at the monthly frequency for the period running from July 1979 to June 2012. Following Gertler and Karadi (2015), the unrestricted VAR includes 12 lags. We then recover the impulse response function of these variables to a monetary policy shock identified relying on the proxy VAR approach where, like Gertler and Karadi (2015), the external instrument is the surprise in the three month ahead futures rate. The identified contractionary monetary policy shock shifts the one-year rate upward, decreases economic activity after one year, raises the excess bond premium persistently, which signals the presence of financial frictions, and leads to a very small negative response of the CPI, without exhibiting any price puzzle (see black lines in Figure 3).

Then we proceed to estimating our D-SVAR by an ALS approach, using the unrestricted VAR as auxiliary model. As in the Blanchard and Quah (1989) example of Section 2.3, the autoregressive matrix R is assumed to be diagonal. We recover four unlabelled structural

[13] In particular, in the case when the number of structural impulse responses exceeds the number of the VAR parameters, the asymptotic distribution of a Wald statistic is degenerated but the bootstrap test can still be implemented following the transformation of the statistic.



The black line is the response to a monetary policy shock, as identified following Gertler and Karadi (2015). The grey line is the response to shock in the D-SVAR with diagonal R matrix. Shaded area represent $\pm$ 1 standard deviation around average D-SVAR response obtained from 1,000 Bootstrap replications. Sample is 1979M7-2012M6.

Figure 3: Responses to Gertler and Karadi (2015) monetary policy shock and D-SVAR's shocks ε_1 and ε_3

shocks that we arbitrarily index $\varepsilon_1, \ldots, \varepsilon_4$. Is one of these shocks the Gertler and Karadi (2015) monetary policy shock? We actually find that ε_1 and ε_3 are two shocks for which Gertler and Karadi (2015) instrument would be valid, as the p-values for non-zero correlation between these two shocks and the instrument are less than 5%, whereas they are above 5% for ε_2 and ε_4. Through the lens of model with four AR(1) shocks, this shows that in Gertler and Karadi (2015), the instrument identifies a shock that is a combination of ε_1 and ε_3, and not a single exogenous monetary policy shocks. Figure 3 plots the responses of the four variables to the two shocks ε_1 and ε_3, together with the responses to the Gertler and Karadi (2015) shock.

The shocks ε_1 gives a very similar response of the one-year government bond rate as compared to Gertler and Karadi (2015). Although the response is more persistent, the response of the one-year government bond rate is also pretty similar. Inspection of the IRF for the other variables reveals that none of these two shocks gives similar responses for the four variables. The credit channel narrative of Gertler and Karadi (2015) requires an increase in the excess bond premium, which is obtained with ε_1, but comes with a persistent increase in industrial production. For the shock ε_3, there is indeed an increase in the one-year rate, a decrease in industrial production but an increase in inflation and a decrease in the excess bond premium, which does not square with the credit channel narrative of Gertler and Karadi (2015).

5.2 Revisiting Christiano et al. (1999) Impact Restrictions SVAR

This section revisits Christiano et al. (1999) who identifies a monetary policy shock by means of impact restrictions.[14] We ask whether the obtained macroeconomic dynamics following a monetary policy shock can be uncovered with a D-SVAR. The D-SVAR makes no assumptions on the impact responses, but only assumes that underlying shocks follow mutually orthogonal AR(1) processes. If Christiano et al. (1999) approach identifies a monetary shock, then we should recover it using our D-SVAR approach —under the assumption that this shock is an independent AR(1) shock.

We estimate a VAR featuring real GDP, the unemployment rate, CPI inflation, commodity price inflation and the federal funds rate, in that order, for the period 1965Q1-2007Q4.[15] The VAR includes four lags. We then recover the impulse response function of these variables to a monetary policy shock identified by impact restrictions as in Christiano et al. (1999): the monetary policy shock corresponds to the shock that shifts the federal funds rate while leaving the other variables unaffected on impact. As well known, the identified contractionary monetary policy shock decreases output with a lag, and increases unemployment after a few quarters. Prices increase for about two years, which is known as the price puzzle, and fall below their initial level after this initial phase.

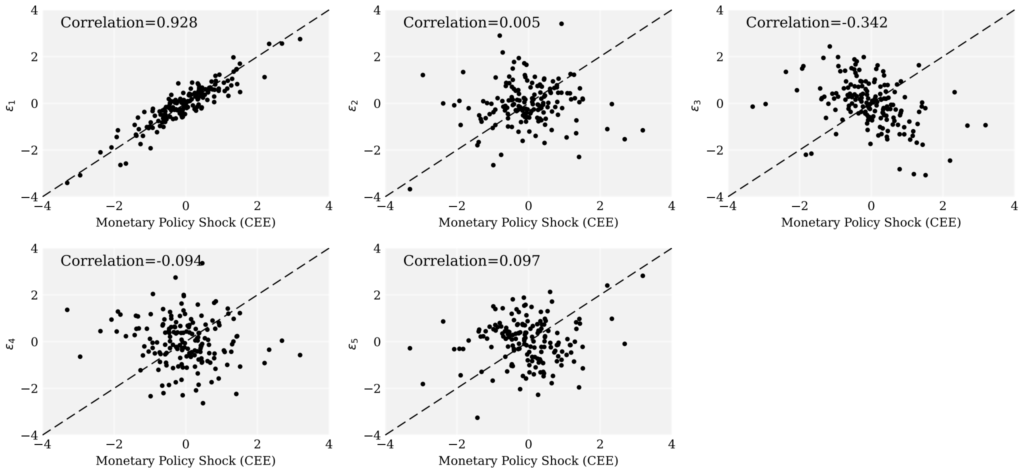
Then we proceed to estimating our D-SVAR by an ALS approach, using the unrestricted VAR as auxiliary model. The J-test indicates that the restrictions imposed by the D-SVAR are not rejected by the data. We recover five unlabelled structural shocks, $\varepsilon_1, \dots, \varepsilon_5$. Is the Christiano et al. (1999) monetary shock one of these shocks? As illustrated in Figure 4, ε_1 is the only shock that exhibits strong positive correlation with Christiano et al. (1999) monetary policy shock shock. Moreover, one cannot reject that this correlation be 1 (p-value=0.986), while one cannot reject that all other shocks are uncorrelated with Christiano et al. (1999) monetary policy shock (p-value$<$0.035).[16]

Figure 5 plots the responses to ε_1 (grey lines), together with the responses to the Christiano et al. (1999) monetary policy shock. The figure indicates that the dynamics following a Christiano et al. (1999) monetary policy shock and those following ε_1 are very similar. Just like for Christiano et al. (1999) monetary policy shock, a tightening of monetary policy leads to a prolonged recession with output (resp. unemployment) reaching it trough after about seven (resp. nine) quarters, and eventually reverting back in the longer run. Accordingly, both output and unemployment exhibit persistent hump shaped dynamics to the shock, just like in the aftermaths of a Christiano et al. (1999) monetary policy shock. Just like in Christiano et al. (1999), prices exhibit a persistent price puzzle, although it is more severe. Such a "price puzzle" is reminiscent of the results in Beaudry et al. (2020), and can be rationalized in a model with a flat Phillips curve and a cost channel.

All in all, the response of the economy is very much in line under the two identification

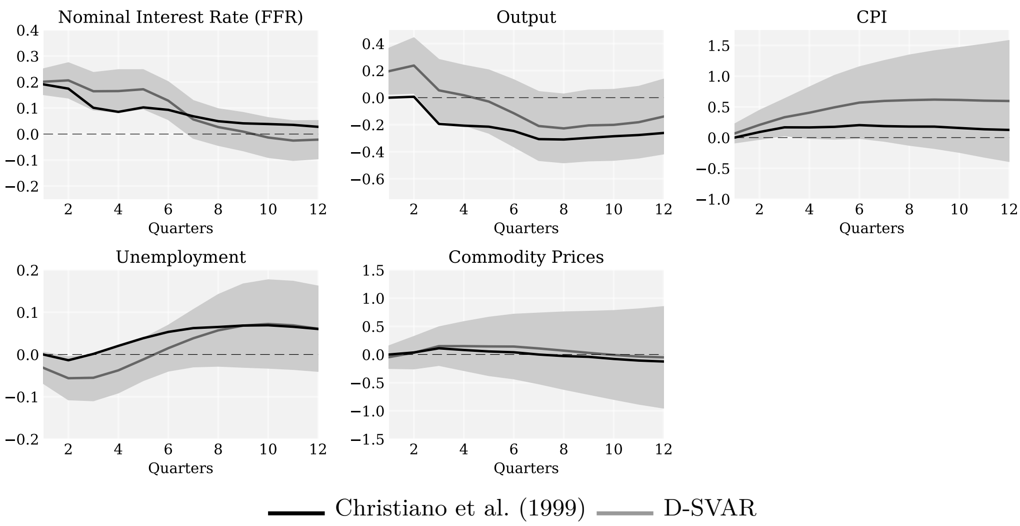
[14]See also Christiano et al. (2005).

[15]We end the sample period in 2007Q4 to avoid dealing with the zero lower bound, which would require an explicit non-linear modelling of the dynamics of the nominal interest rate to account for the presence of an



ε_1 to ε_5 are the structural shocks as obtained with the D-SVAR. Sample is 1965Q1-2007Q4.

Figure 4: Correlation between Christiano et al. (1999) Monetary Shock and D-SVAR Shocks



On the five panel, the black line is the response to a monetary policy shock, as identified following Christiano et al. (1999). The grey line is the response to shock ε_1 in the D-SVAR. Shaded area represent ± 1 standard deviation around average D-SVAR response obtained from 1,000 Bootstrap replications. Sample is 1965Q1-2007Q4.

Figure 5: Responses to Christiano et al. (1999) monetary policy shock and D-SVAR's shock ε_1

Table 2: Test of Zero Impact Restriction (p-values)

	Output	CPI	Unemp.	Com. Price
ε_1	0.2470	0.6587	0.3890	0.8169

ε_1 is the D-SVAR structural shock that is the closest to the Christiano et al. (1999) monetary policy shock. Sample is 1965Q1-2007Q4.

schemes, and hence so are the forecast error variance decomposition (see Table 5 in Appendix .12).[17]

Under the D-SVAR, we can go a step further and test if ε_1 satisfies the zero impact restriction imposed by Christiano et al. (1999). Table 2 reports the p-value associated to the zero impact effect of ε_1 on, respectively, output, the CPI, unemployment and the commodity price. The results are strikingly in favour of the restriction imposed by Christiano et al. (1999) as none of the p-values lies below 25%, well above the 5% standard level. This exercise shows that the D-SVAR recovers a shock that is quite similar to the Christiano et al. (1999) monetary policy one, although there is a more pronounced "price puzzle". It also shows that the Christiano et al. (1999) zero restrictions, which have been criticised as often not compatible with a DSGE model (*e.g.* Carlstrom et al. (2009)), are not rejected by our D-SVAR.

6 Conclusion

In this paper, we have shown that one can identify structural shocks in a SVAR under the identifying assumption that the economy shares the dynamic structure of the vast majority of DSGE models. To put it loosely, if the economy is moved by exogenous variables that follow mutually orthogonal AR(1) processes (or more general specification of the autoregressive matrix), then a D-SVAR will allow for the identification of structural shocks, without the need for zero-impact, long-run or sign restrictions. We have given a formal proof for identification and have shown how to conduct estimation and inference with D-SVAR. We have then applied our methodology to uncover the effects of monetary policy shocks, and shown that D-SVAR give results in line with the most prominent approaches in SVAR the literature, namely proxy-VAR as in Gertler and Karadi (2015) and zero impact restrictions as in Christiano et al. (1999), although the Gertler and Karadi (2015) monetary policy shock cannot be easily thought as a structural shock in a DSGE model.

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occasionally binding constraint (see *e.g.* Mavroeidis (2021)).

[16] See Table 4 in Online Appendix .12.

[17] Figures 10–13 in Appendix .12 report the IRF to ε_2, ε_3, ε_4 and ε_5. Inspection of the figure indicate that none of these shocks is a good candidate to a Christiano et al. (1999) monetary policy shock.

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.1 Preliminaries: Controllability, Observability and Minimality

Consider the ABCD system representation (12) and (13). The system (or the pair (A, B)) is controllable when any state S_t can be driven to the initial state in a finite number of steps for a given input sequence ε_t. A formal definition is given by:

Definition 3 Controllability *: A system is controllable if and only if the controllability matrix* $\mathbb{C} = [B, AB, A^2B, \dots, A^{n_s-1}B] \in \mathbb{R}^{n_s \times n_z n_s}$ *has full row rank i.e., rank*$(\mathbb{C}) = n_s$*.*

If a system is state observable, its present state can be determined from the knowledge of the present and future outputs Y_t and inputs ε_t. A formal definition is given by:

Definition 4 Observability*: A system is observable if and only if the observability matrix* $\mathbb{O}$ *has full column rank, i.e., rank*$(\mathbb{O}) = n_s$*, where*

$$\mathbb{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n_s-1} \end{bmatrix} \in \mathbb{R}^{n_y n_s \times n_s}.$$

Theorem 1 *A state-space representation is minimal if and only if it is controllable and observable.*

Proof : See Antsaklis and Michel (1997), Theorem 3.9, p.395 or Gouriéroux and Monfort (1995), Chap. 8, Property 8.43, p. 282.

.2 Proof of Proposition 1

Consider a general form for the matrix T such that $T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$.

We first show that the relation $\tilde{A}T = TA$ implies that $T_{21} = 0$. Since $\tilde{A}$ is necessarily block upper triangular, $\tilde{R}T_{21} = T_{21}G$ which can be rewritten as:

$$[(I_{n_k} \otimes \tilde{R}) - (G' \otimes I_{n_z})]vec(T_{21}) = 0. \tag{24}$$

Matrix $\tilde{A}$ is *similar* to matrix A which implies that the eigenvalues of $\tilde{A}$ are the same than the eigenvalues of A.[18] Since A is block upper diagonal, for the set of eigenvalues of A denoted $\lambda(A)$, we have $\lambda(A) = \lambda(G) \cup \lambda(R)$. The same property holds for $\tilde{A}$, *i.e.* $\lambda(A) = \lambda(\tilde{G}) \cup \lambda(\tilde{R})$. This implies that the eigenvalues of the matrix $\tilde{R}$ are the same then the eigenvalues of the matrix R. Under Assumption $1''$, R and G share no common eigenvalues, this also holds for $\tilde{R}$ and G. The expression $[(I_{n_k} \otimes \tilde{R}) - (G' \otimes I_{n_z})]$ is then of full rank, Equation (24) holds only for $T_{21} = 0$.[19]

Now, for the block upper triangular matrix $T = \begin{bmatrix} T_{11} & T_{12} \\ 0_{n_z \times n_k} & T_{22} \end{bmatrix}$, the equation $\widetilde{B} = TBU$ gives for the left lower block $I_{n_z} = T_{22}U$. Since U is orthonormal, this implies $T_{22} \equiv U^{-1} = U' = V$ an orthonormal matrix. The matrix T has the following form: $T = \begin{bmatrix} T_{11} & T_{12} \\ 0 & V \end{bmatrix}$. The result for the case where the state variables K_t are observed follows directly.

.3 Proof of Proposition 2

The only admissible orthonormal matrix V such that all elements (i, i) of the diagonal matrix R are identified is $V = I$ when the diagonal element $r_{i,i} \neq r_{j,j}$ for $\forall i \neq j$. For any other orthonormal matrix V, it is easy to verify that the resulting $\tilde{R} = VRV'$ matrix is not diagonal. Since V is a square matrix and is non singular, V corresponds to a *similarity transformation*, the eigenvalues of matrix $\tilde{R}$ are the same as the eigenvalues of matrix R which implies that the diagonal elements of $\tilde{R}$ are the same as the diagonal elements of R. Moreover, for diagonal matrices $\tilde{R}$ and R, the system of equations $\tilde{R}V - VR = 0$ leads to

$$(\tilde{r}_{i,i} - r_{i,i})V_{i,i} = 0 \quad (\text{ for } i = 1, \ldots, n_z), \tag{25}$$

$$(\tilde{r}_{i,i} - r_{j,j})V_{i,j} = 0 \quad (\text{for } i \neq j \text{ and } i, j = 1, \ldots, n_z), \tag{26}$$

where $r_{i,j}$ and $\tilde{r}_{i,j}$ are respectively the element (i, j) of R and $\tilde{R}$.[20] The first set of equation implies that $\tilde{r}_{i,i} = r_{i,i}$ for diagonal elements of V which are different from zero. Since the diagonal

[18] If X is a square matrix and nonsingular, then A and $B = X^{-1}AX$ are *similar* and X is called a *similarity transformation*. If two matrices A and B are *similar*, they have the same eigenvalues, i.e. $\lambda(A) = \lambda(B)$, and the same number of independent eigenvectors but probably not the same eigenvectors (see Golub and Van Loan (2013) p. 349). Moreover, if X is an orthonormal matrix, i.e. $XX' = I$, A and B real matrices and $A = XBX'$, A is said to be real orthogonally similar to B (see Horn and Johnson (2013), p. 94).

[19] If λ is an eigenvalue of A and μ an eigenvalue of B, $\lambda - \mu$ is an eigenvalue of $(I \otimes A) - (B \otimes I)$ and all eigenvalues of $(I \otimes A) - (B \otimes I)$ is on this form. Thus $(I \otimes A) - (B \otimes I)$ has zero as an eigenvalue if and only if A has an eigenvalue λ and B has an eigenvalue μ such that $\lambda - \mu = 0$.

[20] The system of equations $\tilde{R}V - VR = 0$ is of the well known form $AX - BX = C$ in control theory called the Sylvester equation. For $C = 0$, this corresponds to the homogeneous Sylvester equation (see Gantmacher (1959), chap VIII).

elements of R (and then $\tilde{R}$) are different, the second set of equations implies necessarily that $V_{i,j} = 0$ for all $i \neq j$. Matrix V is necessarily diagonal and the only orthonormal matrix which is diagonal is the identity. The result also holds up to changes of sign and/or permutation of the identity matrix.

Assume now that some elements on the diagonal are the same. Denote the multiplicity of similar diagonal elements $r_{i,i}$ by $m(r_{i,i})$. One can first check that all diagonal elements are the same, in which case any orthonormal matrix V is admissible by equations (25) and (26). Now consider that a subgroup of elements has the same value. The elements in the V matrix corresponding to multiple values are not uniquely defined but only up to the post multiplication by an $m(r_{i,i}) \times m(r_{i,i})$ orthogonal matrix. Without lost of generality, suppose that the diagonal elements with the same value are ordered as the first $m(r_{1,1})$ elements on the diagonal and the other elements on the diagonal $r_{j,j}$ are different from $r_{1,1}$, then define the matrix V such that $V = \begin{bmatrix} V_{m(r_{1,1})\times m(r_{1,1})} & 0 \\ 0 & I_{(n_z - m(r_{1,1}))} \end{bmatrix}$, where $V_{m(r_{1,1})\times m(r_{1,1})}$ is an orthonormal matrix. Consequently, there exists an infinity of admissible V matrices such that $V_{m(r_{1,1})\times m(r_{1,1})}$ is orthonormal. This argument can be generalised to more than one diagonal element with multiplicity. A sufficient condition for local identification is therefore that matrix R be diagonal with distinct diagonal elements ($r_{i,i} \neq r_{j,j}$, $\forall i \neq j$).

.4 Proof of Proposition 3.

By $\tilde{R} = VRV'$, one show that the only admissible matrix V which satisfies $\tilde{R}V = VR$ for lower triangular matrices $\tilde{R}$ and R is $V = I_{n_z}$. Since V is of full rank and orthonormal and $\tilde{R} = VRV'$, $\tilde{R}$ and R have the same eigenvalues. Moreover, the eigenvalues of a lower triangular matrix are the elements on the diagonal. By $\tilde{R}V = VR$, we have

$$[(I_{n_z} \otimes \tilde{R}) - (R' \otimes I_{n_z})]vec(V) = 0. \tag{27}$$

This implies that $vec(V)$ belong to the null space of $A = [(I_{n_z} \otimes \tilde{R}) - (R' \otimes I_{n_z})]$. Since $\tilde{R}$ and R have n_z common roots, the null space of A has a dimension equal to n_z. The system of equations (27) can be written more explicitly as:

$$\begin{bmatrix} (\tilde{R}-\mu_1 I) & -r_{2,1}I & -r_{3,1}I & \cdots & -r_{n_z,1}I \\ 0 & (\tilde{R}-\mu_2 I) & -r_{3,2}I & \cdots & -r_{n_z,2}I \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & (\tilde{R}-\mu_{n_z-1}I) & -r_{n_z,n_z-1}I \\ 0 & 0 & \cdots & 0 & (\tilde{R}-\mu_{n_z}I) \end{bmatrix} \begin{bmatrix} V_{[.,1]} \\ V_{[.,2]} \\ \vdots \\ V_{[.,n_z-1]} \\ V_{[.,nz]} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix},$$

with $V_{[.,i]}$ a vector containing the i-th column of matrix V, μ_j for $j = 1, \ldots, n_z$ are the eigenvalues of R and the eigenvalues of $\tilde{R}$ and R are the same which are given by the elements on the diagonal. Under Proposition 3, the eigenvalues μ_j for $j = 1, \ldots, n_z$ are the same as the matrix Λ. Moreover, all sub-matrices $(\tilde{R} - \mu_j I)$ are lower triangular since $\tilde{R}$ is lower triangular and the entire matrix is a block upper triangular matrix.

This system of equations can be solved by forward substitution for the lower triangular block $(\tilde{R} - \mu_{n_z} I)$ to obtain the vector $V_{[.,n_z]}$ and by backward substitution for the upper block diagonal matrices. Thus, the lower triangular form of the system of equations $(\tilde{R}-\mu_{n_z}I)V_{[.,n_z]} = 0$ implies that the elements of $V_{[.,n_z]}$ are equal to zero except the last one V_{n_z,n_z}. Indeed the recursive form of the equations allows to rewrite the system of equations as $\sum_{j=1}^{i} \hat{r}_{i,j}^{n_z} V_{j,n_z} = 0$ for $i = 1, \ldots, n_z$, which imply that $V_{j,n_z} = 0$ except for V_{n_z,n_z} where $\hat{r}_{i,j}^{n_z}$ is the element (i,j) of the sub-matrix $(\tilde{R} - \mu_{n_z} I)$ and $\hat{r}_{n_z,n_z}^{n_z} = 0$ since R and $\tilde{R}$ share the same eigenvalues. According to Proposition 3, this holds under the weaker restriction that only the first sub-diagonal elements are different from zero which implies that $\hat{r}_{1,1}^{n_z} V_{1,n_z} = 0$ and $\sum_{j=i-1}^{i} \hat{r}_{i,j}^{n_z} V_{j,n_z} = 0$ for $i = 2, \ldots, n_z$.

Now, for the next upper block, we have

$$\begin{bmatrix}(\tilde{R} - \mu_{n_z-1}I) & -r_{nz,nz-1}I\end{bmatrix}\begin{bmatrix}V_{[.,n_z-1]}\\ V_{[.,n_z]}\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix},$$

using the preceding result above for $V_{[.,n_z]}$ yields $V_{j,n_z-1} = 0$ for all $j < n_z - 1$. We can continue to solve the following upper blocks by backward substitution to obtain that all elements of the matrix V above the diagonal are equal to zero: $(\tilde{R} - \mu_j I)V_{[.,j]} = 0$ for $j = n_z$ and $(\tilde{R} - \mu_j I)V_{[.,j]} = \sum_{i=j+1}^{n_z}(r_{i,j}I)V_{[.,i]}$ for $j = 1, \ldots n_z - 1$. The resulting matrix V has all elements above the diagonal equal to zero which implies that the only admissible orthonormal matrix is $V = I$ (up to changes of sign and/or permutation of the identity matrix).

.5 Proof of Proposition 4.

Consider the following general 2×2 symmetric matrix $R = \begin{bmatrix}\rho_1 & \tau\\ \tau & \rho_2\end{bmatrix}$ where ρ_1, ρ_2 and τ are real numbers and $\tau \neq 0$. The eigenvalues are the roots of the following characteristic equation:

$$\begin{vmatrix}\rho_1 - \lambda & \tau\\ \tau & \rho_2 - \lambda\end{vmatrix} = (\rho_1 - \lambda)(\rho_2 - \lambda) - \tau^2 = \lambda^2 - \lambda(\rho_1 + \rho_2) + \rho_1\rho_2 - \tau^2.$$

The two roots can be written as $\lambda_{1,2} = \frac{1}{2}\left[\rho_1 + \rho_2 \pm \sqrt{(\rho_1 - \rho_2)^2 + 4\tau^2}\right]$. Since $(\rho_1-\rho_2)^2+4\tau^2 > 0$, the two eigenvalues are necessarily real. By the expressions of λ_1 and λ_2, there exists an infinity of values for ρ_1, ρ_2 and τ that gives the same eigenvalues. In other words, for any orthonormal matrix V and $\tilde{R} = VRV'$, the matrices $\tilde{R}$ and R are *similar* and therefore have the same eigenvalues. The matrix R is then not identifiable.

Now consider the case where the elements on the diagonal have the same value, i.e. $\rho_1 = \rho_2 = \rho$. The two eigenvalues are now given by $= \lambda_{1,2} = \rho \pm \tau$, so that $\lambda_1 + \lambda_2 = 2\rho$ and $\lambda_1 - \lambda_2 = 2\tau$. This implies that there does not exist another 2×2 symmetric matrix $\tilde{R}$ with $\tilde{\rho} \neq \rho$ and/or $\tilde{\tau} \neq \tau$ with $\tilde{\tau} \neq 0$ having the same eigenvalues as matrix R. In this particular case, matrix R is then locally identifiable.

.6 Proof of Proposition 5.

We can consider cases with a block diagonal matrix R with blocks corresponding to the two preceding cases. For example $R = \begin{bmatrix}\tilde{R} & 0\\ 0 & R_2\end{bmatrix}$, where $\tilde{R}$ is a $(n_{z1} \times n_{z1})$ diagonal matrix with different elements on the diagonal and R_2 is any $(n_{z2} \times n_{z2})$ matrix with $z_1 + z_2 = z$. In this case, matrix V has the form $V = \begin{bmatrix} I & 0 \\ 0 & V_{n_{z2} \times n_{z2}} \end{bmatrix}$, where $V_{n_{z2} \times n_{z2}}$ is an orthonormal matrix. The first z_1 exogenous processes are then locally identified. Matrix $\tilde{R}$ could be also lower triangular.

Additional Online Material

.7 Presence of Unobserved State Variables

We investigate the the case of a vector X_t consisting of both observed, Y_t, and unobserved state variables, K_t (*i.e.* $X_t = (K_t, Y_t)'$). This implies, in particular, that Π_{yk} is not necessarily equal to the identity matrix, and, more importantly, that $\Pi_{yz} \neq 0$. The standard RBC model studied in McGrattan (2010) is a typical example of such a situation as the capital stock may not be directly observed (or at least not without measurement errors) by the econometrician. In that case, the model takes the form of System (2).

As for the previous case, we impose that Y_t has the same dimension as Z_t ($n_y = n_z$). We consider the case where Z_t consists of at least two elements and where there are not more state variables than elements in Z_t. Without loss of generality we take K_t to be of the same order as Z_t by allowing G to be possibly less than full rank.[21] Moreover, we assume that F, Π_{yk} and R are full rank. Note that this does not preclude Π_{yz} from being less than full rank and even possibly zero.

Making use of these assumptions, the dynamics of Y_t can be expressed as

$$Y_t = C_1 Y_{t-1} + C_2 Z_t + C_3 Z_{t-1} \tag{28}$$

where $C_1 = \Pi_{yk} G \Pi_{yk}^{-1}$, $C_2 = \Pi_{yk} F + \Pi_{yz}$ and $C_3 = -\Pi_{yk} G \Pi_{yk}^{-1} \Pi_{yz}$. The issue is then whether, when R is diagonal (or lower triangular), C_1, C_2, C_3 and R can be identified.[22]

Making use of the Z_t process in (28), it is easy to show that the vector of observed variables Y_t follows a VARMA(2,1) process of the form

$$Y_t = \left(DRD^{-1} + C_1\right) Y_{t-1} - DRD^{-1} C_1 Y_{t-2} + C_2 \varepsilon_t + (D - DRD^{-1} C_2)\varepsilon_{t-1}.$$

where $D \equiv C_3 + C_2 R$. Key for this result is the fact that D be invertible, which is guaranteed by the full rank assumption we placed on G, Π_{yk} and R.

As in the previous case, counting the number of coefficients to uncover and the number of moments the VARMA(2,1) structure offers, we recover that imposing a lower triangular structure on R provides us with the right number of restrictions. Note that this does not generically guarantees identification, as the system that needs to be solved features a quadratic term implying that a pair of solutions generally arise. However, as long as R is sparser than a triangular matrix, the system features more equations than unknowns. The order condition is then clearly satisfied (in fact, it is over-identified). As before, checking the rank condition is non trivial, and we follow another strategy in the paper to formally prove identification.

[21] This is without loss of generality if, when the number of state variables is less than the dimension of Z_t, we add non state variables in the first equation allowing G to potentially have columns of zeroes.

[22] Note that while G, F and Π_{yk} cannot (in general) be identified separately. This is however not an issue as far as the identification of the structural impulse responses is concerned as all that is needed is the identification of R and Cs. Indeed, we will identify $\Pi_{yk} G \Pi_{yk}^{-1}$, $\Pi_{yk} F$, Π_{yz} and R.

.8 Partial Identification: An Example

Let us consider the textbook 3-equation NK model developed in Example 1 in the main text and let us introduce a monetary policy shock, labeled $z_{3,t}$, that exogenously shifts the interest rate set by the monetary authority —*i.e.* $i_t = \alpha_\pi \pi_t + z_{3,t}$. To ease exposition, we assume that the two shocks $z_{1,t}$ and $z_{2,t}$ are not serially correlated, while third one $z_{3,t}$ is. As will be clear later, the two serially uncorrelated shocks cannot be separately identified, whereas the third one one can be identified as long as it is serially correlated. Just as in Example 1, assuming the Taylor principle holds, the model can be solved forward and yields the state-space representation

$$X_t = FZ_t,$$

where F is a (3×3) matrix. The vector $X_t = (y_t, \pi_t, i_t)'$ gathers the three endogenous variables and $Z_t = (z_{1,t}, z_{2,t}, z_{3,t})'$ is the vector of the three structural shocks. Given our assumptions on the dynamics of the shocks, Z_t evolves as

$$Z_t = RZ_{t-1} + \varepsilon_t \quad \text{where} \quad R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \rho \end{bmatrix} \quad \text{and} \quad \varepsilon_t = \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \\ \varepsilon_{3,t} \end{bmatrix}.$$

ε_t is a zero mean weak white noise and $E(\varepsilon_t \varepsilon_t') = I_3$.

Similarly to Example 1, the moving average representation of output (likewise for the inflation rate and the nominal interest rate) writes

$$y_t = f_{11}\varepsilon_{1,t} + f_{12}\varepsilon_{2,t} + f_{13}\sum_{i=0}^{\infty} \rho^i \varepsilon_{3,t-i},$$

from which we get the output auto-covariance functions as

$$\gamma_y(0) = f_{11}^2 + f_{12}^2 + \frac{f_{13}^2}{1-\rho^2} \quad \text{when} \quad h = 0,$$
$$\gamma_y(h) = \frac{f_{13}^2 \rho^h}{1-\rho^2} \quad \text{when} \quad h > 0.$$

The persistence parameter ρ can be directly identified by computing the ratio $\gamma_y(h+1)/\gamma_y(h)$ for $h > 0$, which is free from any parameter f_{13}. Given ρ, the direct observation of $\gamma_y(h)$, for any $h > 0$, allows to recover parameter f_{13} (up to a sign term). In other words, the effect of a monetary policy shock on output is identified. Applying the same procedure on inflation and the nominal interest allows for the identification of f_{23} and f_{33} (up to a sign term). It is therefore possible to identify the monetary transmission mechanism for all variables in the D-SVAR model.

Note that, on the contrary, the knowledge of $\gamma_y(0)$ only helps identify the sum $f_{11}^2 + f_{12}^2$, not f_{11} and f_{12} separately, In other words, the effect of two remaining shocks $\varepsilon_{1,t}$ and $\varepsilon_{2,t}$ cannot be separately identified. The reason for this result is that the dynamics of these two shocks do not bring any information to disentangle them. This result obviously extends to the inflation rate and the nominal interest rate. This example simply illustrates that one shock can be identified in so far as it displays a dynamic structure that differs from the other shocks in the economy.

.9 Data Appendix

.9.1 Data from Section 2.3

- **Real GDP** is measured as the ratio of Gross Domestic Product in value (Table 1.1.5 from BEA) divided by the GDP price index (Table 1.1.4 from BEA), and is expressed in per capita term by dividing by the Civilian non-institutional population from 16 years of age and older residing in the 50 states and the District of Columbia (`CNP16OV` in the Federal Reserve DataBase (FRED, `http://fred.stlouisfed.org`)).

- The **unemployment gap** is measured as the difference between the average unemployment rate over a quarter (`UNRATE` from FRED) and the natural rate of unemployment — *i.e.* the rate of unemployment arising from all sources except fluctuations in aggregate demand (`NROU` from FRED).

.9.2 Data from Section 5.1

The data from Section 5.1 are borrowed from Gertler and Karadi (2015) and are downloadable from `http://doi.org/10.3886/E114082V1`

- The **price level** is measured by the Consumer Price Index for all urban consumers (`CPIAUCSL` from FRED, All Items in U.S. City Average)

- **Economic activity** is measured by the industrial production index (`INDPRO` from FRED)

- The **nominal interest rate** is measured by the Market Yield on U.S. Treasury Securities at 1-Year Constant Maturity (`GS1` fom FRED)

- The **credit spread** is measured by the Gilchrist and Zakrajšek (2012) excess bond premium.

- The **external instrument** corresponds to the three month ahead monthly fed funds futures (`FF4` from Gertler and Karadi (2015))

The price level and the economic activity index are both expressed in logs prior to estimation.

.9.3 Data from Section 5.2

- The **nominal interest rate** is measured by the Effective Federal Funds Rate (`DFF` from FRED).

- **Output** is measured as the Real Gross Domestic Product expressed in Billions of Chained 2012 Dollars (`GDPC1` from FRED, Quarterly, Seasonally Adjusted Annual Rate).

- The **price level** is measured by the Consumer Price Index for all items for the United States (`CPALTT01USM661S` from FRED, Index 2015=100, Quarterly, Seasonally Adjusted).

- The **unemployment rate** corresponds to the quarterly average of the monthly unemployment rate in the US (`UNRATE` from FRED, Percent, Quarterly, Seasonally Adjusted)
- The **commodity price** is the Producer Price Index of all commodities (`PPIACO` from FRED, Index 1982=100, Quarterly).

Output, the Price level and the commodity price are all transformed by applying the log function prior to estimation.

.10 The Bivariate D-SVAR of Section 2.3

As is well-know (see, *e.g.* Fernald (2007)), Blanchard and Quah (1989) identification is sensitive to the long-run properties of the variables since it requires the estimation of the spectral density of, at least, one variable at frequency 0 —an object which is usually hard to estimate and at best very imprecise. This makes this approach quite non robust in case of trend breaks. The Great Financial Crisis (GFC) of 2008 presents the econometrician with such a challenge, as illustrated in Figure 6.

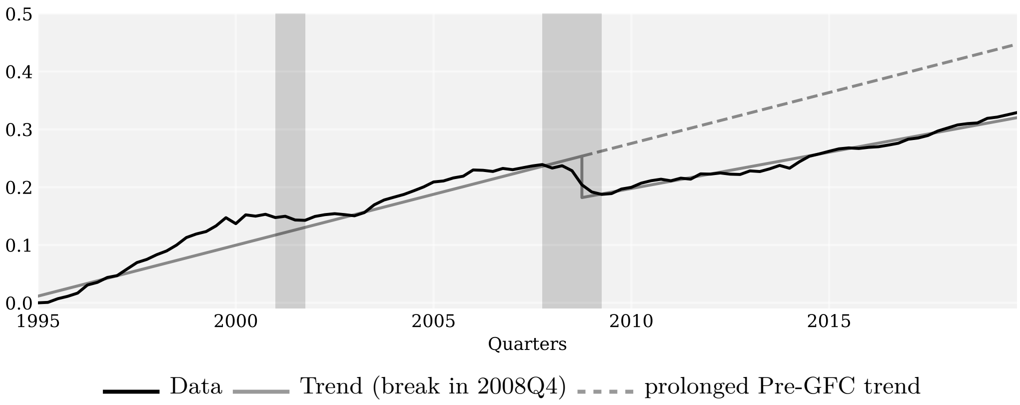
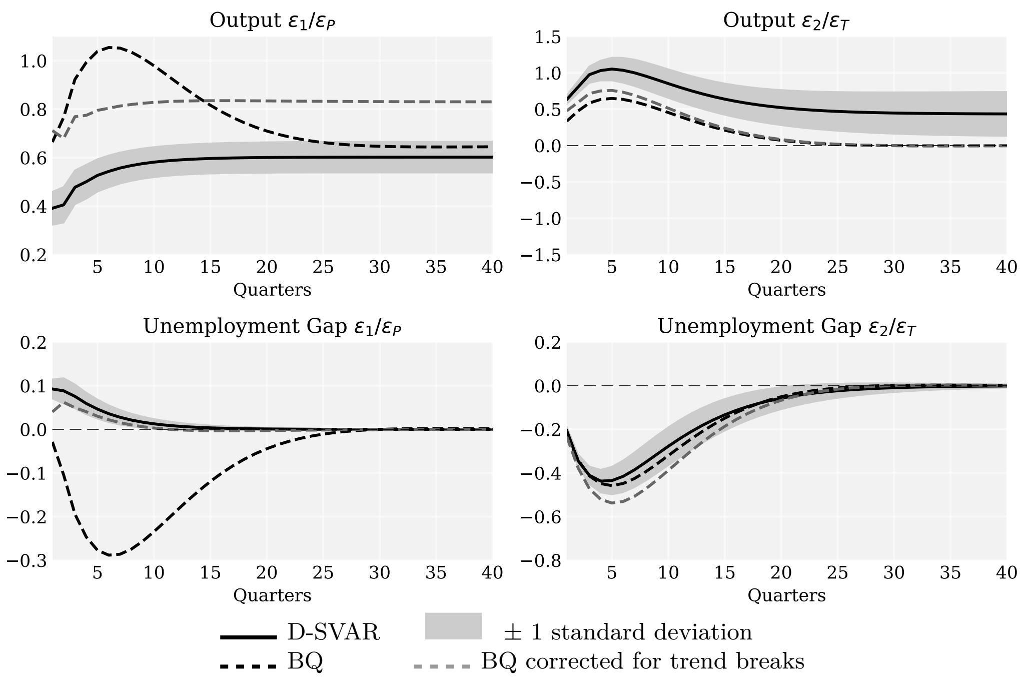


Figure 6: Output per Capita (in logs)



Sample is 1960Q1-20197Q4. y is the real GDP, u is the unemployment rate gap. Estimation is done with $(\Delta y, u)$ using two lags. The grey area represents 68% confidence bands obtained from 1,000 Bootstrap replications.

Figure 7: Impulse Response Functions: 1960Q1-2019Q4

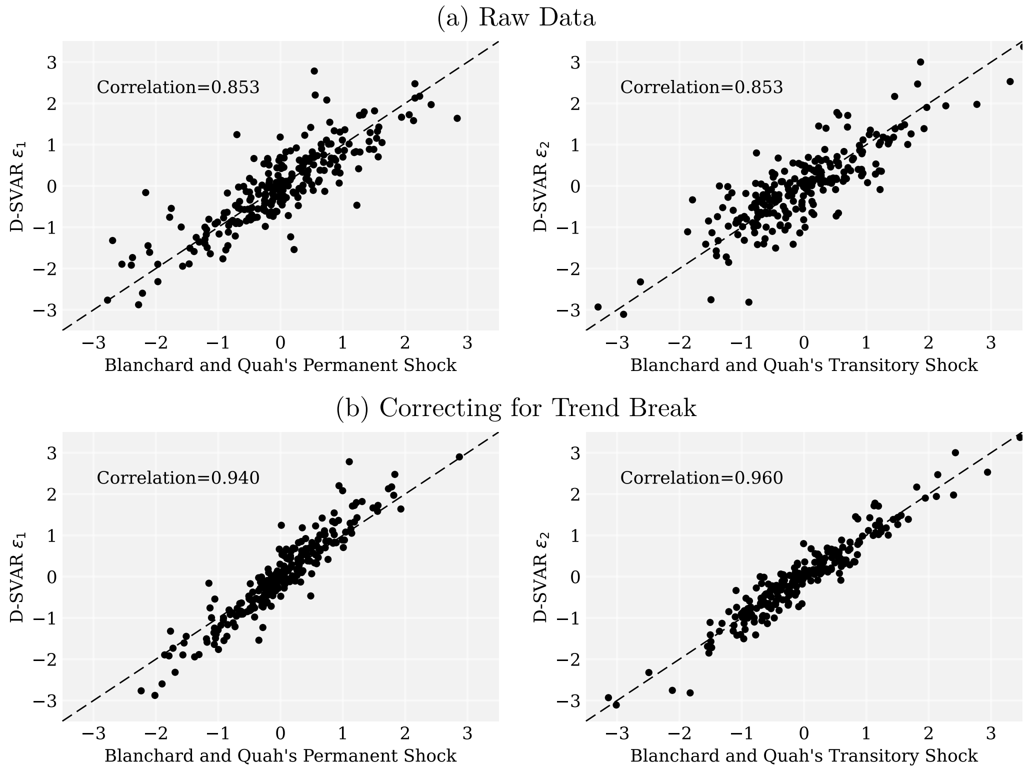
As illustrated in Figure 7 and Table 3, when the sample period is extended up to 2019Q4, the dynamic implications of the two shocks ε_1 and ε_2 essentially remain unaffected both for output and the unemployment gap. ε_1 can still be interpreted as the "permanent" shock, ε_2 as

Table 3: Forecast Error Variance Decomposition, Extended sample 1960Q1–2019Q4

	Output				Unemployment gap			
Horizon	ε_1	ε_2	ε_P	ε_T	ε_1	ε_2	ε_P	ε_T
1	27.7	72.3	79.7	20.3	16.8	83.2	1.6	98.4
4	20.6	79.4	72.6	27.4	4.7	95.3	17.0	83.0
8	21.9	78.1	73.7	26.3	2.6	97.4	25.0	75.0
20	32.6	67.4	81.5	18.5	2.0	98.0	28.5	71.5
∞	65.8	34.2	100.0	0.0	2.0	98.0	28.6	71.4

Sample is 1960Q1-2019Q4. Estimation is done with $(\Delta y, u)$ using two lags, where y is the real GDP and u is the unemployment rate gap. ε_1 and ε_2 correspond to the D-SVAR, ε_P and ε_T to Blanchard and Quah (1989).

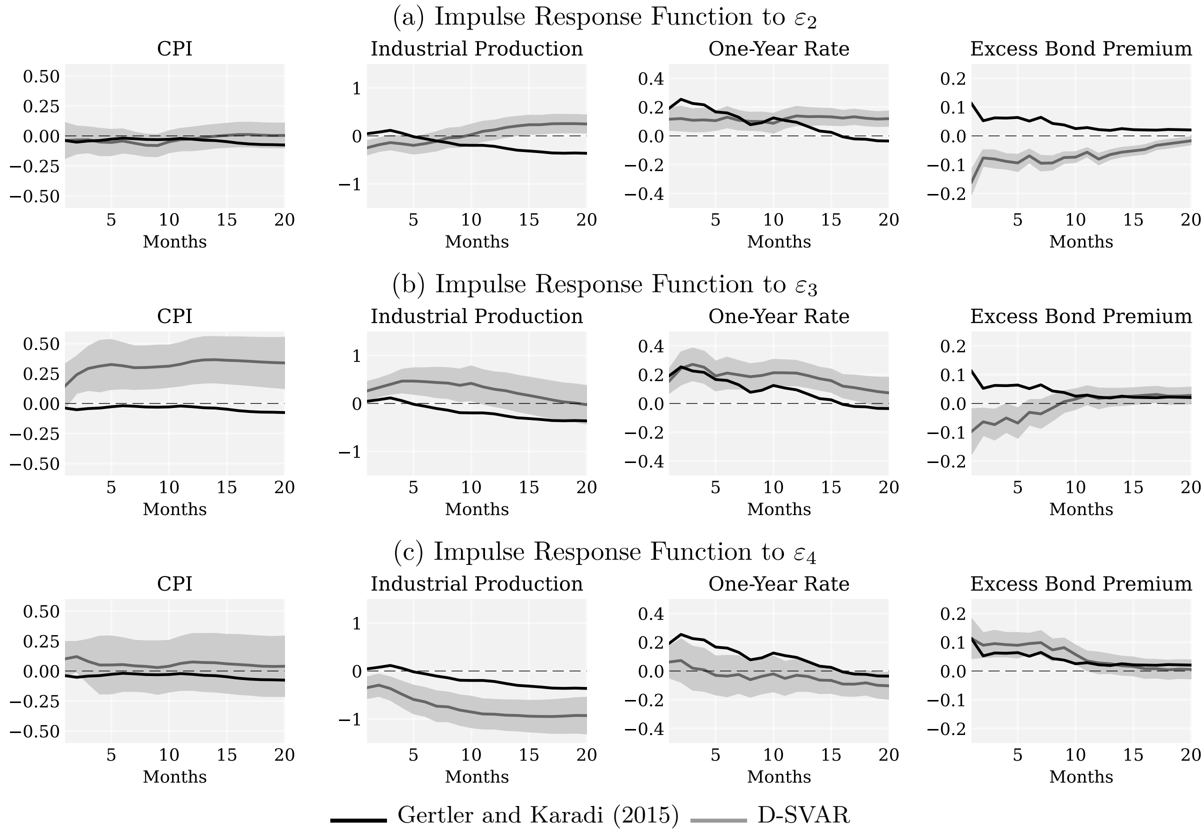
the "transitory" shock, and the shape of the response of both output and the unemployment gap to both shocks and the associated forecast error variance decomposition are essentially unaffected by the shift to the extended sample. Things are different in so far as the BQ decomposition is concerned. The black dash line in Figure 7 reports the dynamics of output and the unemployment gap to both the permanent and transitory shocks, as recovered by BQ's identification. As evident by comparing Figures 1 and 7 the response of both variables to the permanent shock are largely affected by the extension of the sample. More strikingly, this extension leads to a reversal in the respective contribution of the shocks to output dynamics: the permanent shock now becomes the main driver of output dynamics (see Panel (b) of Table 1). As soon as output dynamics is corrected for the trend break, the responses to a permanent shock resembles those to ε_1 in our VAR. Figure 8 shows that, correcting for trend breaks leads to an increase in the correlation between BQ shocks and the shocks as identified by the D-SVAR (from 0.85 to 0.95). The gains from the dynamic identification are then clear: by not directly relying on long-run restrictions, the D-SVAR is much less sensitive to breaks in variables featuring a trend.



Sample is 1960Q1-2019Q4. Estimation is done with $(\Delta y, u)$ using two lags, where y is the real GDP and u is the unemployment rate gap.

Figure 8: Correlation between D-SVAR and BQ shocks: 1960Q1-2019Q4

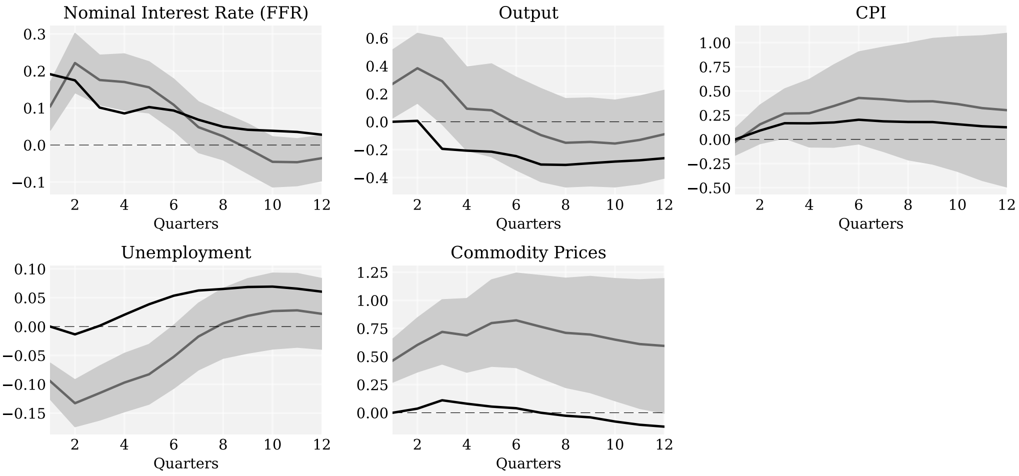
.11 Additional Material for Section 5.1



On the four panels, the black line is the response to a monetary policy shock, as identified following Gertler and Karadi (2015). The grey line is the response to shock in the D-SVAR. Shaded area represent ± 1 standard deviation around average D-SVAR response obtained from 1,000 Bootstrap replications. Sample is 1979M7-2012M6.

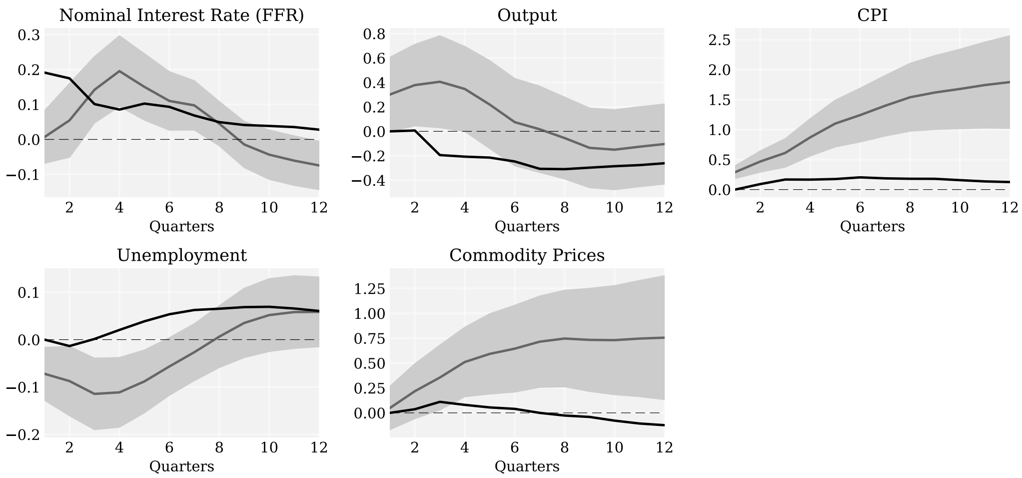
Figure 9: Responses to Gertler and Karadi (2015) monetary policy shock and D-SVAR's shocks with Lower Triangular R Matrix (ε_2–ε_4)

.12 Additional Material for Section 5.2



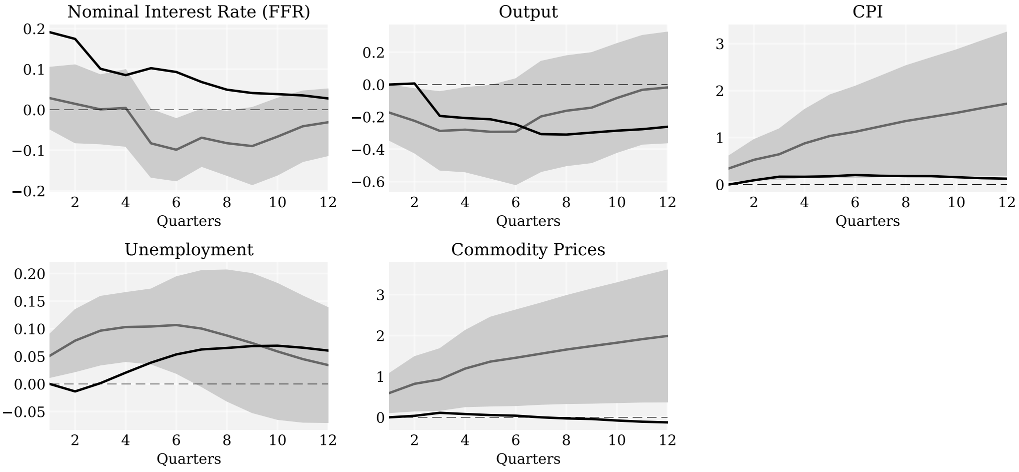
Black line: response to a monetary policy shock, as identified following Christiano et al. (1999). Grey line: response to shock ε_4 in the D-SVAR. Shaded area: ± 1 standard deviation around average D-SVAR response obtained from 1,000 Bootstrap replications. Sample is 1965Q1-2007Q4.

Figure 10: Responses to Christiano et al. (1999) monetary policy shock and D-SVAR's shock ε_2



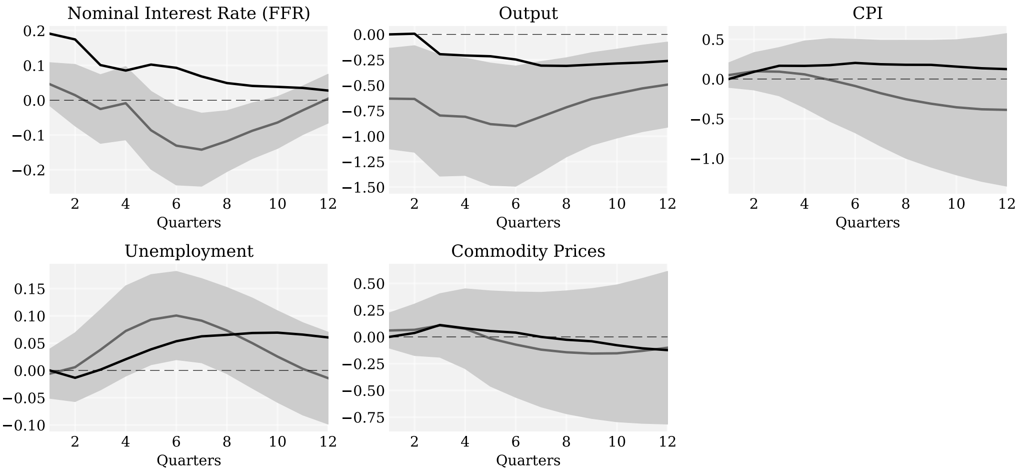
Black line: response to a monetary policy shock, as identified following Christiano et al. (1999). Grey line: response to shock ε_4 in the D-SVAR. Shaded area: ± 1 standard deviation around average D-SVAR response obtained from 1,000 Bootstrap replications. Sample is 1965Q1-2007Q4.

Figure 11: Responses to Christiano et al. (1999) monetary policy shock and D-SVAR's shock ε_3



Black line: response to a monetary policy shock, as identified following Christiano et al. (1999). Grey line: response to shock ε_4 in the D-SVAR. Shaded area: $\pm$ 1 standard deviation around average D-SVAR response obtained from 1,000 Bootstrap replications. Sample is 1965Q1-2007Q4.

Figure 12: Responses to Christiano et al. (1999) monetary policy shock and D-SVAR's shock ε_4



Black line: response to a monetary policy shock, as identified following Christiano et al. (1999). Grey line: response to shock ε_4 in the D-SVAR. Shaded area: $\pm$ 1 standard deviation around average D-SVAR response obtained from 1,000 Bootstrap replications. Sample is 1965Q1-2007Q4.

Figure 13: Responses to Christiano et al. (1999) monetary policy shock and D-SVAR's shock ε_5

Table 4: Correlation between CEE's MP Shock and DSVAR Shocks

	ε_1	ε_2	ε_3	ε_4	ε_5
Corr	0.928	-0.005	-0.342	0.094	0.097
p(0)	0.000	0.986	0.353	0.695	0.652
p(1)	0.665	0.013	0.015	0.035	0.011

Note: Corr reports the correlation between the CEE Monetary Policy Shock and each of the shock as identified by our DSVAR. p(0) (resp. p(1)) reports the p-value associated to the Wald test of a unit (resp. zero) correlation between the CEE Monetary Policy Shock and each of the shock as identified by our DSVAR. This p-value is obtained by means of 1,000 bootstrap, so as to accommodate for the fact that both shocks are generated by an estimated VAR model and are hence contaminated by the uncertainty surrounding the estimated coefficients of the VAR.

Table 5: Forecast Error Variance Decomposition of (in %)

	Christiano et al. (1999)	D-SVAR				
Horizon	ε_R	ε_1	ε_2	ε_3	ε_4	ε_5
Nominal Interest Rate						
1	66.79	74.37	20.16	0.09	1.50	3.89
4	26.03	42.62	37.11	19.00	0.32	0.94
8	17.74	34.46	29.15	19.75	5.36	11.28
20	13.39	31.30	27.11	21.18	7.42	12.99
Output						
1	0.00	6.08	11.72	14.39	4.77	63.04
4	2.47	3.01	9.65	15.92	7.38	64.04
8	5.47	3.22	5.47	8.92	7.38	75.01
20	5.95	3.48	4.12	6.23	4.81	81.36
Price Index (CPI)						
1	0.00	2.16	0.30	40.50	55.86	1.17
4	1.80	9.12	4.80	40.62	44.78	0.68
8	1.04	8.83	4.36	46.40	39.72	0.70
20	0.23	5.44	1.66	46.05	45.27	1.58
Unemployment						
1	0.00	5.61	50.38	29.17	14.64	0.20
4	0.46	6.51	37.46	29.12	21.81	5.11
8	5.70	5.93	25.66	21.69	29.76	16.97
20	6.47	13.22	20.87	22.49	28.28	15.14
Commodity Price						
1	0.00	0.37	37.40	0.42	61.19	0.62
4	0.37	0.89	29.19	8.12	61.34	0.47
8	0.13	0.56	21.06	12.05	65.98	0.35
20	0.49	0.14	9.86	12.86	76.95	0.19

In this table we compare the variance decomposition as obtained by Christiano et al. (1999) for their monetary policy shocks and for the five shocks of the D-SVAR. Sample is 1965Q1-2007Q4.

.13 A Two-Country VAR

In this section, we consider the modelling of a 2-country VAR featuring the log-difference of US GDP and residual of the cointegration relationship (1,-0.63) between the Euro Area and US real GDP for the 1995Q1-2019Q4 period as reported by OECD (`https://stats.oecd.org/`, VPVOBARSA, US Dollars, volume estimates at fixed PPP, seasonally adjusted). In this example, we illustrate how the D-SVAR allows to recover a shock structure à' la Backus et al. (1992) involving dynamic symmetric spillovers. More precisely the shock process is assumed to take the form

$$Z_t \equiv \begin{pmatrix} z_t^{us} \\ z_t^{can} \end{pmatrix} = \begin{bmatrix} \rho & \nu \\ \nu & \rho \end{bmatrix} Z_{t-1} + \varepsilon_t \text{ with } \varepsilon_t \rightsquigarrow N(0_2, I_2),$$

where ν captures the dynamic spillovers. Both the AIC, BIC and Hannan-Quinn information criteria led us to select a VAR(2) specification. The D-SVAR identification then leads to a value of $\rho = 0.43$ and $\nu = 0.20$. The loading matrix B then takes the form

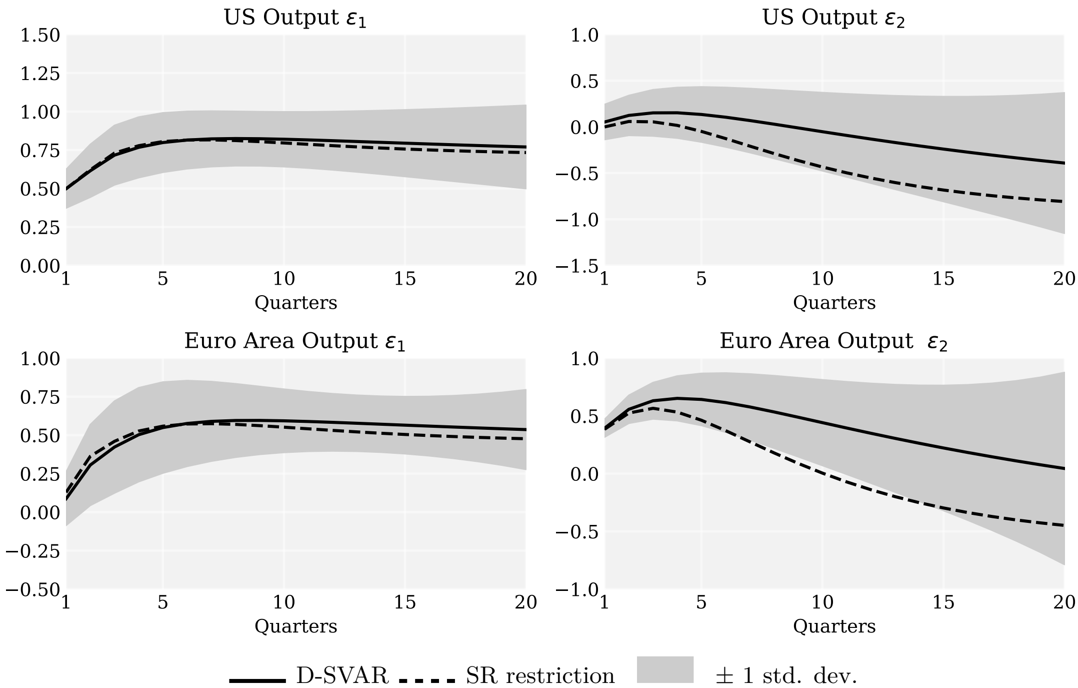
$$B = \begin{bmatrix} 0.498 & 0.053 \\ -0.229 & 0.360 \end{bmatrix}$$

The forecast error variance decomposition of levels is reported in Table 6.. While the US shock explain essentially all of US GDP, it accounts for less than 5%the Euro volatility in the very short-run and 56% at the 20 quarters horizon. In other words, in the very short-run, the US and Euro Area economies are essentially insulated from each other, while the shocks are transmitted in the medium run. Figure 14 reports the IRF of US and Euro GDP to both shocks. These IRF confirm and illustrate the broad picture conveyed by forecast error volatility decomposition: US and Euro GDP only responds to their respective shocks on impact, and are essentially insulated from exogenous developments in the other economy in the very short-run. This is actually slightly contrasts with an identification of the US shock as the only shock that affects US GDP on impact. In that latter case, the US shock is transmitted faster to the Euro Area: the US shock accounts for about 10% of the Euro output volatility on impact and about 40% after 1 year (30% in our case). In the longer run, the Cholesky decomposition indicates that while the US economy is essentially not affected by Euro shocks, US shocks account for 60% of GDP volatility in the Euro Area.

Table 6: Forecast Error Decomposition

	US GDP		Euro Area GDP	
Horizon	ε_t^{us}	ε_t^{euro}	ε_t^{us}	ε_t^{euro}
D-SVAR				
1	98.9	1.1	4.6	95.4
4	96.4	3.6	29.3	70.7
8	97.8	2.2	40.9	59.1
20	93.6	6.4	60.8	39.2
Short-Run Restriction				
1	100.0	0.0	10.0	90.0
4	99.6	0.4	38.3	61.7
8	96.7	3.3	56.5	43.5
20	68.7	31.3	67.0	33.0

The variance decomposition is obtained from a bivariate D-SVAR or a SVAR with a short-run restriction. Variables are the log-difference of US GDP and the residual of the cointegration relationship (1,-0.63) between the Euro Area and US real GDP. Sample is 1995Q1-2019Q4.



These IRF are obtained from a bivariate D-SVAR or a SVAR with a short-run restriction. Variables are the log-difference of US GDP and the residual of the cointegration relationship (1,-0.63) between the Euro Area and US real GDP. Sample is 1995Q1-2019Q4.

Figure 14: Impulse Response Functions: US vs Euro Area

A Assessing the Ability of DSVAR to Recover Theoretical Shocks

In this section, we assess the ability of the D-SVAR approach to recover the IRF of a theoretical model. More precisely, we will consider successively a canonical New Keynesian model and a Real Business Cycle model and use each of them as Data Generating Process to simulate artificial time series. We then estimate a D-SVAR on those series and ask whether it recovers the IRF to the true structural shocks.

A.1 A New Keynesian Model

A.1.1 Description of the Model

The set up is standard. The economy is populated by a large number of identical infinitely–lived households and economy consists of two sectors: one producing intermediate goods and the other final goods. The intermediate good is produced with labor and the final good with intermediate goods.

The Household: Household preferences are characterised by the lifetime utility function:[23]

$$\mathbb{E}_t \sum_{\tau=0}^{\infty} \beta^{\tau} \omega_t \left(\frac{(c_{t+\tau} - h\bar{c}_{t+\tau-1})^{1-\gamma}}{1-\gamma} - \vartheta \frac{n_{t+\tau}^{1+\varphi}}{1+\varphi} \right) \tag{29}$$

where $0 < \beta < 1$ is a constant discount factor, c denotes consumption and n labor.

In each and every period, the representative household faces a budget constraint of the form

$$B_t + P_t c_t \quad \leq \quad R_{t-1} B_{t-1} + \Pi_t + P_t w_t n_t \tag{30}$$

where B_t are nominal bonds acquired during period t, P_t is the nominal price of the final good, R_{t-1} is the nominal interest rate, w_t denotes the real wage. The household consumes c_t and supplies n_t units of labor and claims the profits, Π_t, earned by the firms. ω_t will act as a demand shock and can be interpreted as a premium shock.

The first order conditions lead to

$$\vartheta n_t^{\varphi} = (c_t - h\bar{c}_{t-1})^{-\gamma} w_t \tag{31}$$

$$\omega_t (c_t - h\bar{c}_{t-1})^{-\gamma} = \beta R_t \mathbb{E}_t \left[\frac{\omega_{t+1}(c_{t+1} - h\bar{c}_t)^{-\gamma}}{\pi_{t+1}} \right] \tag{32}$$

where $\pi_t = P_t / P_{t-1}$ denotes the gross inflation rate.

Final sector: The final good is produced by combining intermediate goods. This process is described by the following CES function

$$y_t = \left(\int_0^1 y_t(i)^{\eta_t} \mathrm{d}i \right)^{\frac{1}{\eta_t}} \tag{33}$$

[23] $\mathbb{E}_t(.)$ denotes mathematical conditional expectations. Expectations are conditional on information available at the beginning of period t.

where $\eta_t \in (-\infty, 1)$. η_t determines the elasticity of substitution between the various inputs, which will be modelled as a stochastic process and will appear as a cost push shock in the New Keynesian Phillips curve. The producers in this sector are assumed to behave competitively and to determine their demand for each good, $y_t(i)$, $i \in (0,1)$ by maximising the static profit equation

$$\max_{\{X_t(i)\}_{i\in(0,1)}} P_t y_t - \int_0^1 P_t(i) y_t(i) \mathrm{d}i \tag{34}$$

subject to (33), where $P_t(i)$ denotes the price of intermediate good i. This yields demand functions of the form:

$$y_t(i) = \left(\frac{P_t(i)}{P_t}\right)^{\frac{1}{\eta_t - 1}} y_t \tag{35}$$

and the following general price index

$$P_t = \left(\int_0^1 P_t(i)^{\frac{\eta_t}{\eta_t - 1}} \mathrm{d}i\right)^{\frac{\eta_t - 1}{\eta_t}} \tag{36}$$

The final good may be used for consumption — private or public — and investment purposes.

Intermediate Good Producers: Each firm i, $i \in (0,1)$, produces an intermediate good by means of capital and labor according to a constant returns–to–scale technology, represented by the production function

$$y_t(i) = n_t(i) \tag{37}$$

where $n_t(i)$ denotes the labor input used by firm i in the production process. a_t is an exogenous stationary stochastic technology shock. Assuming that each firm i operates under perfect competition in the input markets, the firm determines its production plan so as to minimise its total cost

$$\min_{\{h_t(i)\}} P_t w_t n_t(i)$$

subject to (37). This yields to the following expression for total costs:

$$P_t s_t y_t(i)$$

where the real marginal cost, s_t, is simply given by w_t.

Intermediate goods producers are monopolistically competitive, and therefore set prices for the good they produce. We follow Calvo [1983] in assuming that firms set their prices for a stochastic number of periods. In each and every period, a firm either gets the chance to adjust its price (an event occurring with probability $1-\alpha$) or it does not. When the firm does not reset its price, it just applies steady state inflation to the price it charged in the last period such that $P_t(i) = \pi_{t-1}^{\zeta} \overline{\pi}^{1-\zeta} P_{t-1}(i)$. When it gets a chance to do it, firm i resets its price, $\widetilde{P}_t(i)$, in period t in order to maximise its expected discounted profit flow this new price will generate. In period t, the profit is given by $\Pi(\widetilde{P}_t(i))$. In period $t+1$, either the firm resets its price, such that it will get $\Pi(\widetilde{P}_{t+1}(i))$ with probability α, or it does not and its $t+1$ profit will be $\Pi(X_{t,t+1}\widetilde{P}_t(i))$

with probability $(1-\alpha)$. Likewise in $t+2$. Expected profit flow generated by setting $\widetilde{P}_t(i)$ in period t writes

$$\max_{\widetilde{P}_t(i)} \mathbb{E}_t \sum_{\tau=0}^{\infty} \Phi_{t,t+\tau} \alpha^{\tau-1} \Pi(X_{t,t+\tau} \widetilde{P}_t(i))$$

subject to the total demand it faces:

$$y_t(i) = \left(\frac{\widetilde{P}_t(i)}{P_t}\right)^{\frac{1}{\eta-1}} y_t$$

where $X_{t,t+1} = \pi_t^{\zeta} \overline{\pi}^{1-\zeta} X_{t-1,t}$ and $\Pi(X_{t,t+\tau}\widetilde{P}_t(i)) = \left(X_{t,t+\tau}\widetilde{P}_t(i) - P_{t+\tau}s_{t+\tau}\right) y_{t+\tau}(i)$. $\Phi_{t+\tau}$ is an appropriate discount factor related to the way the household values future as opposed to current consumption, such that

$$\Phi_{t,t+\tau} \propto \beta^{\tau} \frac{\Lambda_{t+\tau}}{\Lambda_t} \text{ where } \Lambda_{t+\tau} \equiv \omega_{t+\tau}(c_{t+\tau} - h\overline{c}_{t+\tau-1})^{-\gamma}$$

This leads to the price setting equation

$$\mathbb{E}_t \left[\sum_{\tau=0}^{\infty} (\beta\alpha)^{\tau} \frac{\Lambda_{t,t+\tau}}{\eta_{t+\tau} - 1} \left(\eta_{t+\tau} X_{t,t+\tau} \widetilde{P}_t(i) - P_{t+\tau} s_{t+\tau}\right) y_{t+\tau}(i) \right] = 0 \tag{38}$$

From the definition of the aggregate price (36) and the Calvo fairy assumption, the aggregate price level may be expressed as

$$P_t = \left(\sum_{j=0}^{\infty} (1-\alpha)\alpha^j (X_{t-j,t}\widetilde{P}_{t-j})^{\frac{\eta_t}{\eta_t - 1}} \right)^{\frac{\eta_t - 1}{\eta_t}} \tag{39}$$

Monetary Authorities: Monetary authorities are assumed to follow a Taylor rule of the form (in log-linear deviations from deterministic steady state)

$$i_t = \rho_r i_{t-1} + (1-\rho_r)(\phi_\pi \pi_t + \phi_y y_t) + \epsilon_{i,t}$$

where $|\rho_r| < 1$ and $\phi_y, \phi_\pi > 0$.

Equilibrium: An equilibrium of this economy is a sequence of prices $\{\mathcal{P}_t\}_{t=0}^{\infty} = \{w_t, , P_t, R_t, \widetilde{P}_t\}_{t=0}^{\infty}$ and a sequence of quantities $\{\mathcal{Q}_t\}_{t=0}^{\infty} = \{\{\mathcal{Q}_t^H\}_{t=0}^{\infty}, \{\mathcal{Q}_t^F\}_{t=0}^{\infty}\}$ with

$$\begin{aligned} \{\mathcal{Q}_t^H\}_{t=0}^{\infty} &= \{c_t, B_t, n_t\}_{t=0}^{\infty} \\ \{\mathcal{Q}_t^F\}_{t=0}^{\infty} &= \{y_t, y_t(i), n_t(i); i \in (0,1)\}_{t=0}^{\infty} \end{aligned}$$

such that:

(i) given a sequence of prices $\{\mathcal{P}_t\}_{t=0}^{\infty}$ and a sequence of shocks, $\{\mathcal{Q}_t^H\}_{t=0}^{\infty}$ is a solution to the representative household's problem;

(ii) given a sequence of prices $\{\mathcal{P}_t\}_{t=0}^{\infty}$ and a sequence of shocks, $\{\mathcal{Q}_t^F\}_{t=0}^{\infty}$ is a solution to the representative firms' problem;

(iii) given a sequence of quantities $\{\mathcal{Q}_t\}_{t=0}^{\infty}$ and a sequence of shocks, $\{\mathcal{P}_t\}_{t=0}^{\infty}$ clears the markets. In particular, we have $\int_0^1 y_t(i)\mathrm{d}i = c_t$ and $\int_0^1 n_t(i)\mathrm{d}i = n_t$.

(iv) Prices satisfy (38) and (39).

Log-linearisation of the equilibrium around the deterministic steady state gives rise to the following three (log-)linearised equations

$$y_t = \frac{h}{1+h}y_{t-1} + \frac{1}{1+h}\mathbb{E}_t[y_{t+1}] - \frac{1-h}{\gamma(1+h)}(i_t - \mathbb{E}_t[\pi_{t+1}]) + z_{d,t} \tag{40}$$

$$\pi_t = \frac{\zeta}{1+\beta\zeta}\pi_{t-1} + \frac{\beta}{1+\beta\zeta}\mathbb{E}_t[\pi_{t+1}] + \frac{(1-\alpha)(1-\beta\alpha)}{\alpha(1+\beta\zeta)}(\gamma+\varphi)y_t + z_{s,t} \tag{41}$$

$$i_t = \rho_r i_{t-1} + (1-\rho_r)(\phi_\pi \pi_t + \phi_y y_t) + z_{r,t} \tag{42}$$

with $z_{j,t} = \rho_j z_{j,t-1} + \varepsilon_{j,t}$, where $\varepsilon_{j,t} \rightsquigarrow N(0,\sigma_j^2)$ with $j \in \{d,s,r\}$. As long as the Taylor principle holds, the solution of the model admits a state space representation of the form

$$\begin{pmatrix} y_t \\ \pi_t \\ i_t \end{pmatrix} = G(\theta)\begin{pmatrix} y_{t-1} \\ \pi_{t-1} \\ i_{t-1} \end{pmatrix} + F(\theta)\begin{pmatrix} z_{1,t} \\ z_{2,t} \\ z_{3,t} \end{pmatrix} \text{ where } \begin{pmatrix} z_{1,t} \\ z_{2,t} \\ z_{3,t} \end{pmatrix} = R(\theta)\begin{pmatrix} z_{1,t-1} \\ z_{2,t-1} \\ z_{3,t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \\ \varepsilon_{3,t} \end{pmatrix}$$

where θ collects all the parameters of the model. The state space representation rewrites as a VAR(2) as[24]

$$\begin{pmatrix} y_t \\ \pi_t \\ i_t \end{pmatrix} = (G(\theta) + F(\theta)R(\theta)F(\theta)^{-1})\begin{pmatrix} y_{t-1} \\ \pi_{t-1} \\ i_{t-1} \end{pmatrix} - F(\theta)R(\theta)F(\theta)^{-1}G(\theta)\begin{pmatrix} y_{t-2} \\ \pi_{t-2} \\ i_{t-2} \end{pmatrix} + F(\theta)\begin{pmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \\ \varepsilon_{3,t} \end{pmatrix}$$

We first estimate the model by a Bayesian Maximum Likelihood Estimation method on US data excluding the Zero Lower Bound period (1960Q1-2007Q4). Output gap is measured by the negative of the gap between the unemployment rate and the long-run natural rate of unemployment. The inflation rate is measured by the annualised quarterly change in GDP deflator, and the annualised Effective Federal Fund Rate is used as a measure of the nominal interest rate. Table 7 reports the priors used during the estimation as well as the posterior mode, mean and 90% high probability density intervals obtained from a MCMC algorithm.

The dynamic properties of the estimated model, as reported in Figure 15, are in line with the conventional wisdom. A demand shock (left panel of Figure 15) rises output, inflation and the nominal interest rate. A cost push shock (center panel) increases inflation, reduces output and the Fed reacts by raising the policy rate. Finally, the hike in the interest rate that follows a contractionary monetary policy shock, depresses economic activity and reduces inflation.

A.1.2 Assessing the D-SVAR Approach Using the NK Model as the DGP

We run the following Monte-Carlo experiment. We use the estimated NK model as the DGP for output, inflation and the nominal interest rate and simulate it 1,000 times over the 1,000,000.

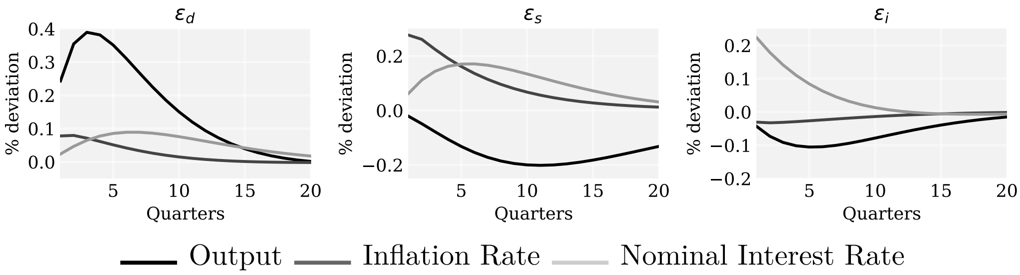
[24] The VAR representation obtains thanks to the observability of all variables in the state space representation of the solution. Appendix A.2 considers a standard Real Business Cycle model in which capital cannot be observed by the econometrician. In that case, the solution does not admit a VAR representation, but a VARMA. We illustrate that most of the results we will discuss in this section extend to the VARMA case.

Table 7: New Keynesian Model, Priors and Posteriors

	Priors			Posteriors		
	distribution	Mean	std. dev.	Mode	Mean	90% HPDI
h	Beta	0.50	0.10	0.9082	0.8974	[0.8552;0.9437]
σ	Gamma	1.00	0.50	2.2951	2.5200	[1.4757;3.5418]
ϕ	Gamma	1.00	0.50	0.5217	0.7183	[0.1540;1.2722]
α	Beta	0.50	0.10	0.9074	0.9077	[0.8749;0.9414]
ζ	Beta	0.50	0.15	0.0872	0.1073	[0.0375;0.1729]
ρ_i	Beta	0.50	0.20	0.8133	0.8072	[0.7615;0.8562]
ϕ_π	Normal	1.50	0.25	1.4542	1.4719	[1.1928;1.7406]
ϕ_y	Normal	0.10	0.05	0.1391	0.1382	[0.0666;0.2112]
ρ_d	Beta	0.50	0.20	0.5869	0.5867	[0.4980;0.6728]
ρ_s	Beta	0.50	0.20	0.8981	0.8909	[0.8364;0.9471]
ρ_r	Beta	0.50	0.20	0.2310	0.2454	[0.1305;0.3630]
σ_d	Inv. Gamma	0.10	2.00	0.0598	0.0609	[0.0480;0.0738]
σ_s	Inv. Gamma	0.10	2.00	0.0388	0.0421	[0.0308;0.0533]
σ_r	Inv. Gamma	0.10	2.00	0.2268	0.2301	[0.2099;0.2499]

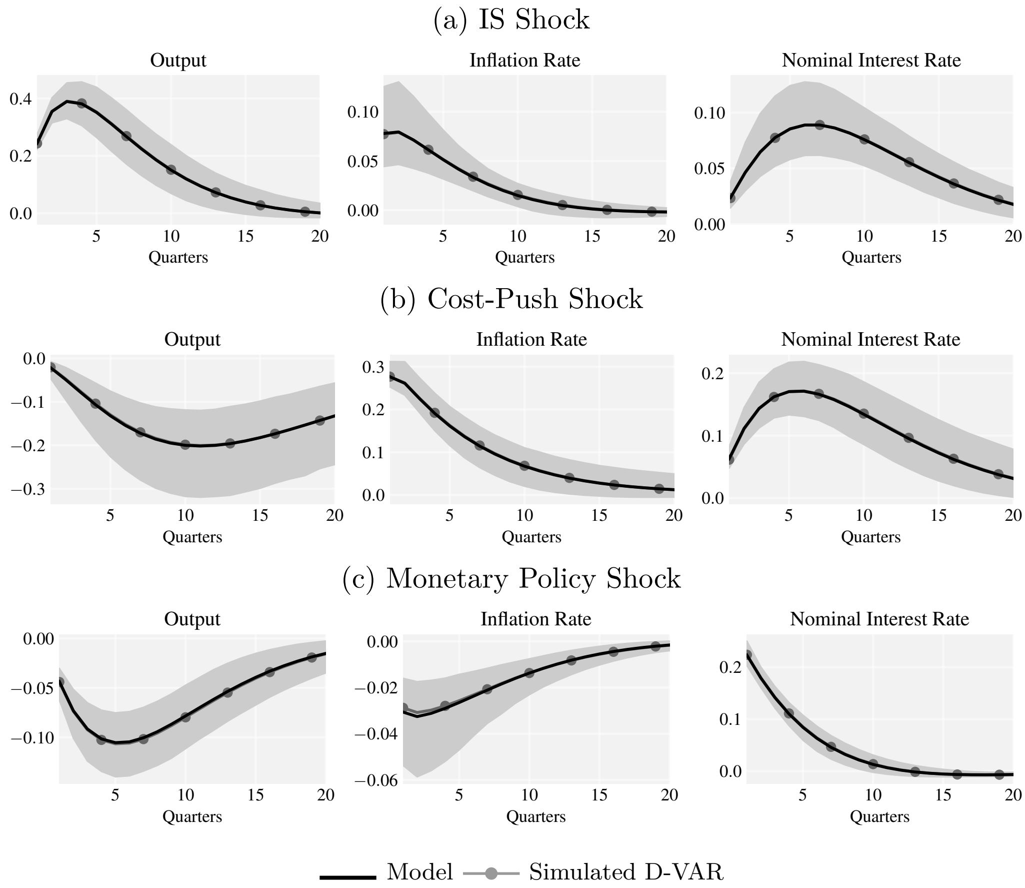
Notes: The estimated model is the New Keynesian (40)–(42). Sample is 1960Q1-2007Q4. Estimation is done with minus the unemployment gap, GDP deflator inflation and the Federal fund rate. Posterior distribution obtained from MCMC using 2 chains of 200,000 draws each.

Figure 15: Impulse responses, Estimated New-Keynesian Model



We report here the average theoretical impulse response function (IRF) of output, inflation and the nominal interest rate across the MCMC chains. The estimated model is the New Keynesian (40)–(42). Sample is 1960Q1-2007Q4. Estimation is done with minus the unemployment gap ("output"), GDP deflator inflation and the Federal fund rate.

Figure 16: Impulse responses, NK model vs D-SVAR Estimated on Simulated Data



The estimated model is a D-SVAR with two lags, data are generated by the estimated New Keynesian model. We report the average of 1,000 estimations of length 1,000,000. The shaded area corresponds to the 95% confidence band of each IRF in the theoretical model, as obtained from the MCMC chains.

For each simulation, we estimate the following unrestricted VAR

$$\begin{pmatrix} y_t \\ \pi_t \\ i_t \end{pmatrix} = \Phi_1 \begin{pmatrix} y_{t-1} \\ \pi_{t-1} \\ i_{t-1} \end{pmatrix} + \Phi_2 \begin{pmatrix} y_{t-2} \\ \pi_{t-2} \\ i_{t-2} \end{pmatrix} + \begin{pmatrix} u_{1,t} \\ u_{2,t} \\ u_{3,t} \end{pmatrix}$$

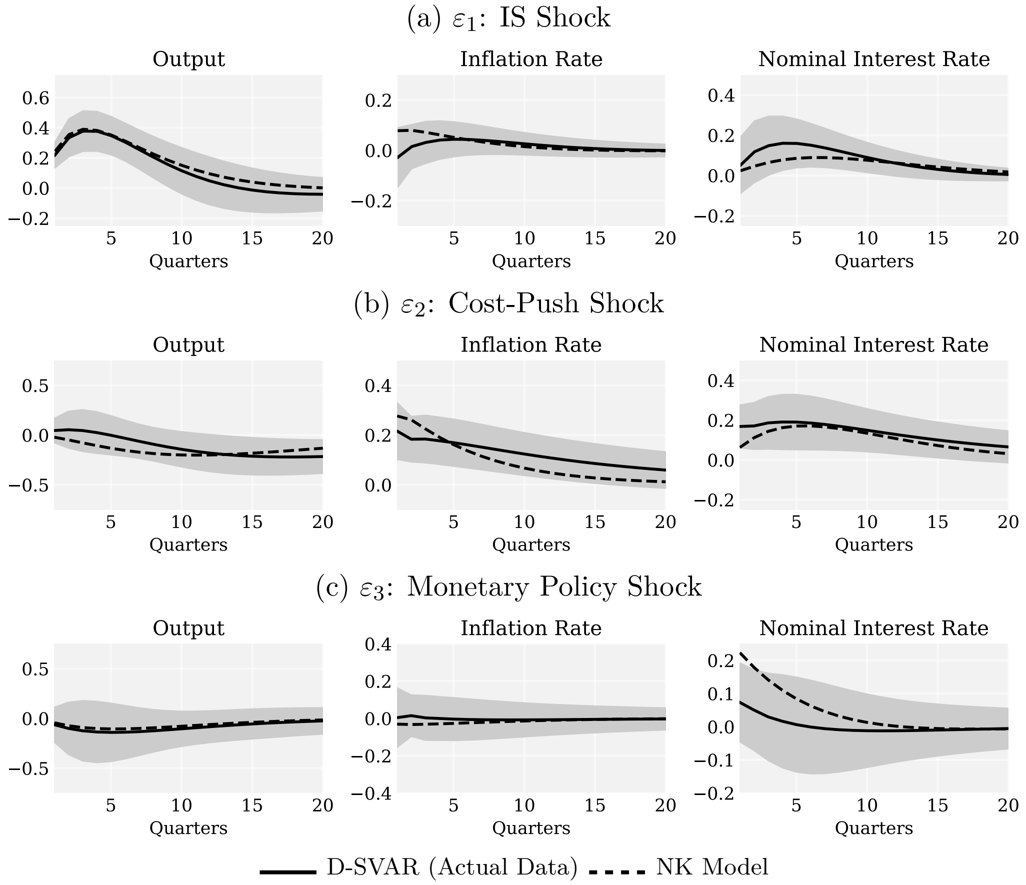
and then recover the D-SVAR representation

$$\begin{pmatrix} y_t \\ \pi_t \\ i_t \end{pmatrix} = G \begin{pmatrix} y_{t-1} \\ \pi_{t-1} \\ i_{t-1} \end{pmatrix} + F \begin{pmatrix} z_{1,t} \\ z_{2,t} \\ z_{3,t} \end{pmatrix} \quad \text{where} \quad \begin{pmatrix} z_{1,t} \\ z_{2,t} \\ z_{3,t} \end{pmatrix} = R \begin{pmatrix} z_{1,t-1} \\ z_{2,t-1} \\ z_{3,t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \\ \varepsilon_{3,t} \end{pmatrix}$$

using the ALS estimation method. We then compute the response of each variable to each shock. Figure 16 reports for each shock the New Keynesian model average theoretical IRF of output, inflation and the nominal interest rate across the MCMC chains (the ones already reported on Figure 16) (plain dark line) alongside the average IRF as recovered from the simulated D-SVAR (bullet plain line). The shaded area corresponds to the 95% confidence band of each IRF in the theoretical model, as obtained from the MCMC chains.

Note that the three shocks are unlabelled in the D-SVAR, so we order them by minimizing the distance between the model structural shock and each D-SVAR shock. Inspection of the

Figure 17: Impulse responses, New Keynesian Model vs D-SVAR Estimated on Actual Data



The estimated model are the New Keynesian (40)–(42) and a D-SVAR with two lags, using actual data over the sample 1960Q1-2007Q4. Estimation is done with minus the unemployment gap ("output"), GDP deflator inflation and the Federal fund rate. The shaded area correspond to the D-SVAR 95% confidence band as obtained from bootstrap (1,000 draws).

figure suggests that the D-SVAR allows to recover exactly the three structural shocks: the IRF are on top of each other.

A.1.3 Assessing the Cross-Equation Restrictions of the NK Model

We first estimate an unrestricted VAR on the same data we used to estimate the New Keynesian model, and use it as an auxiliary model to recover the D-SVAR representation by ALS. Figure 17 then reports the impulse response functions of output, inflation and the interest rate as obtained from the D-SVAR along with their 95% confidence bands. Again, we order the responses by similarity with the theoretical ones. Note that this set of impulse responses ought to differ from those of the estimated New Keynesian model. Indeed, although the two models share the same dynamic structure (same variables, same lags, same processes for the latent exogenous variables), the New Keynesian model estimation imposes more cross-equation restrictions than in the D-SVAR. Strikingly, the responses, as recovered from our D-SVAR (plain line), show similarities with the theoretical responses (dashed line).

Comparison of the theoretical and data IRF shows that responses to a demand shock are similarly estimated by the New Keynesian model and the D-SVAR, although inflation response to the ε_1 shock is negative on impact. There is a shock in the D-SVAR, ε_2, that does increase inflation and the nominal interest rate and decreases output after five periods, as does the cost push in the New Keynesian model. But the short-run response of output is positive in the D-SVAR, which is not the typical prediction of a New Keynesian model. As far as the monetary shock is concerned, the ε_3 shock in the D-SVAR is indeed increasing the nominal interest rate and decreasing output, but it increases inflation in the short-run, while inflation response is always negative in the estimated New Keynesian model. Such a "price puzzle" is reminiscent of the results in Beaudry et al. (2020), and can be rationalized in a model with a flat Phillips curve and a cost channel. Note that scale of the responses to that shock also speaks in favor of a flat Phillips curve: the response of output is of the same magnitude in the New Keynesian and D-SVAR model, while the response of the nominal interest is more than twice as small in the D-SVAR, while inflation is barely moving. Overall, much of the joint dynamics estimated with the New Keynesian model can already be uncovered with the D-SVAR, without having to impose all the cross-equation restrictions of the DSGE.

A.2 A Real Business Cycle Model

We consider a real business cycle model featuring a *catching up with the Joneses* mechanism and real frictions on the capital accumulation process that take the form of investment adjustment costs. The problem of the Central planner takes the form

$$\begin{aligned}
&\max \mathbb{E}_t \left[\sum_{\tau=0}^{+\infty} \beta^\tau \left(\log(C_{t+\tau} - b\overline{C}_{t+\tau-1}) - \vartheta_{t+\tau}^{-1} \frac{h_{t+\tau}^{1+\nu}}{1+\nu}\right)\right] \\
&C_t + I_t = A_t K_t^\alpha (\Gamma_t h_t)^{1-\alpha} \\
&K_{t+1} = \zeta_t I_t \left(1 - \Phi\left(\frac{I_t}{I_{t-1}}\right)\right) + (1-\delta) K_t
\end{aligned}$$

where $\beta \in (0,1)$ denotes the discount factor, $b \in (0,1)$ governs habit persistence, $\nu > 0$ is the inverse of the Frish elasticity, $\alpha \in (0,1)$ is capital elasticity and the function $\Phi(\cdot)$ is strictly increasing and convex and satisfies $\Phi(\gamma) = \Phi'(\gamma) = 0$. $\mathbb{E}_t[\cdot]$ denotes the conditional expectation operator. Furthermore, $\varphi \equiv \Phi''(\gamma)\gamma > 0$ governs the importance of investment adjustment costs. Γ_t denotes exogenous technological progress that evolves deterministically as $\Gamma_t = \gamma \Gamma_{t-1}$, $\gamma > 1$. Finally ϑ_t, A_t and ζ_t denote respectively a labor wedge, a technology and an investment specific shock, which are all assumed to follow a stationary AR(1) process of the form

$$\log(X_t) = \rho_X \log(X_{t-1}) + \varepsilon_t^X \text{ for } x \in \{\vartheta, A, \zeta\}$$

where $|\rho_X| < 1$ and $\varepsilon^X \rightsquigarrow N(0, \sigma_X^2)$. The (deflated for growth, $x_t = X_t/\Gamma_t$) optimal allocation of this economy is then characterised by the set of equations

$$\begin{aligned}
h_t^\nu &= (1-\alpha)\frac{y_t}{h_t}\frac{\gamma\vartheta_t}{\gamma c_t - b c_{t-1}} \\
1 &= \frac{\beta}{\gamma}\mathbb{E}_t\left[\frac{\gamma c_t - b c_{t-1}}{\gamma c_{t+1} - b c_t}\left(\alpha\frac{y_{t+1}}{k_{t+1}} + (1-\delta) q_{t+1}\right)\right] \\
1 &= \zeta_t q_t \left(1 - \Phi\left(\gamma\frac{i_t}{i_{t-1}}\right) - \Phi'\left(\gamma\frac{i_t}{i_{t-1}}\right)\gamma\frac{i_t}{i_{t-1}}\right) + \frac{\beta}{\gamma}\mathbb{E}_t\left[\frac{\gamma c_t - b c_{t-1}}{\gamma c_{t+1} - b c_t}\zeta_{t+1}\Phi'\left(\gamma\frac{i_{t+1}}{i_t}\right)\gamma\frac{i_{t+1}}{i_t}\right] \\
y_t &= A_t k_t^\alpha h_t^{1-\alpha} \\
y_t &= c_t + i_t \\
\gamma k_{t+1} &= \zeta_t i_t \left(1 - \Phi\left(\gamma\frac{i_t}{i_{t-1}}\right)\right) + (1-\delta) k_t
\end{aligned}$$

where lowercase variable x denotes the deflated for growth variable X ($x_t = X_t/\Gamma_t$) for any $X \in \{Y, C, I, K\}$. The solution of a log-linearised version of the optimal allocation admits the state space representation:

$$\begin{aligned}
Y_t &= \Pi_X X_t + \Pi_Z Z_t \\
X_{t+1} &= M_X X_t + M_Z Z_t \\
Z_{t+1} &= R Z_t + \varepsilon_t
\end{aligned}$$

where $X_t = (\widehat{k}_t, \widehat{c}_{t-1}, \widehat{i}_{t-1})'$, $Y_t = (\widehat{c}_t, \widehat{y}_t, \widehat{i}_t, \widehat{h}_t, \widehat{q}_t)$ and $Z_t = (\widehat{a}_t, \widehat{\zeta}_t, \widehat{\vartheta}_t)'$ and $\varepsilon_t = (\varepsilon_t^A, \varepsilon_t^\zeta, \varepsilon_t^\vartheta)'$. As usual in the literature $\widehat{x}_t$ denotes the log-deviation of variable x from its deterministic steady

Table 8: Parametrisation

Preferences		
β	Discount Factor	0.990
b	Habit Persistence	0.650
ν	Inv. Frish Elasticity	1.000
Technology		
α	Capital Elasticity	0.330
φ	Investment Adjustment Costs	2.500
δ	Depreciation Rate	0.025
γ	Gross Rate of Growth	1.004
Shock Persistence		
ρ_A	Technology Shock	0.950
ρ_ζ	Investment Specific Shock	0.810
ρ_ϑ	Labor Wedge Shock	0.940
Shock Volatility (in %)		
ρ_A	Technology Shock	0.700
ρ_ζ	Investment Specific Shock	2.000
ρ_ϑ	Labor Wedge Shock	0.800

state. We then assess the ability of the dynamic identification technique developed in the main text to recover the dynamics of the "true" structural model. The main difference from the New-Keynesian model investigated in the text is that the structural model features a latent variable unobserved by the econometrician —e.g. the capital stock.

In order to perform this assessment we parametrise the model by borrowing values for the structural parameters from the RBC literature (see Table 8). The parameters pertaining to the investment specific shock are directly borrowed from Justiniano et al. (2011). Those pertaining to the labor wedge shock are taken from Kascha and Mertens (2009).[25] We then run the following Monte-Carlo experiment. We use the RBC model as the DGP for output, investment and hours worked and simulate the estimated model 10,000 times over 250 periods. For each simulation, we estimate the following VARMA

$$\begin{pmatrix} y_t \\ i_t \\ h_t \end{pmatrix} = \Phi_1 \begin{pmatrix} y_{t-1} \\ i_{t-1} \\ h_{t-1} \end{pmatrix} + \Phi_2 \begin{pmatrix} y_{t-2} \\ i_{t-2} \\ h_{t-2} \end{pmatrix} + \begin{pmatrix} u_{1,t} \\ u_{2,t} \\ u_{3,t} \end{pmatrix} + \Theta_1 \begin{pmatrix} u_{1,t-1} \\ u_{2,t-1} \\ u_{3,t-1} \end{pmatrix}$$

and use it as auxiliary model to recover the D-SVARMA representation by ALS as

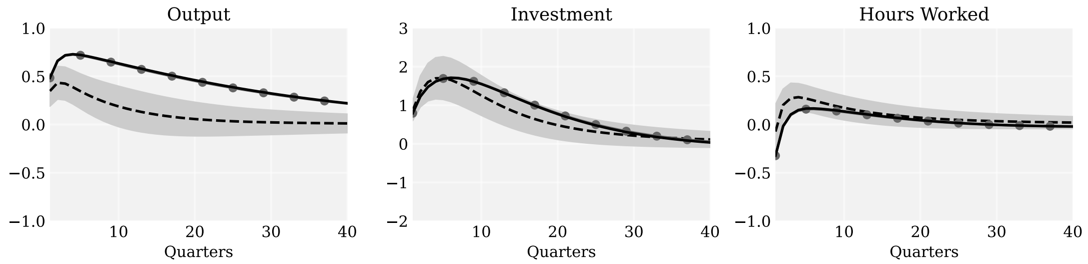
$$\begin{pmatrix} y_t \\ \pi_t \\ i_t \end{pmatrix} = G \begin{pmatrix} y_{t-1} \\ \pi_{t-1} \\ i_{t-1} \end{pmatrix} + F_0 \begin{pmatrix} z_{1,t} \\ z_{2,t} \\ z_{3,t} \end{pmatrix} + F_1 \begin{pmatrix} z_{1,t-1} \\ z_{2,t-1} \\ z_{3,t-1} \end{pmatrix} \text{ where } \begin{pmatrix} z_{1,t} \\ z_{2,t} \\ z_{3,t} \end{pmatrix} = R \begin{pmatrix} z_{1,t-1} \\ z_{2,t-1} \\ z_{3,t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \\ \varepsilon_{3,t} \end{pmatrix}$$

We then compute the response of each variable to each shock. Figure 18 reports for each shock the theoretical IRF of output, investment and hours worked (plain dark line) alongside the average IRF as recovered from the simulated D-SVAR (dashed line). The shaded area corresponds to the 68% confidence band of each IRF in the simulated model.

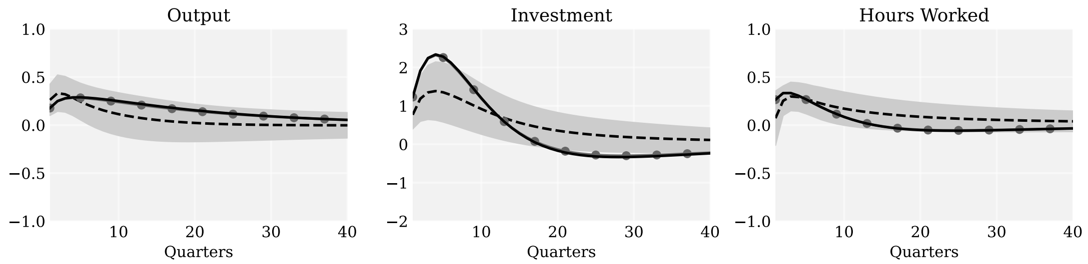
[25] Note that the model does not pretend to be an accurate representation of a specific economy, but is used as a *reasonable* DGP to conduct our assessment exercise.

Figure 18: Impulse responses, RBC model vs Simulated D-SVARMA

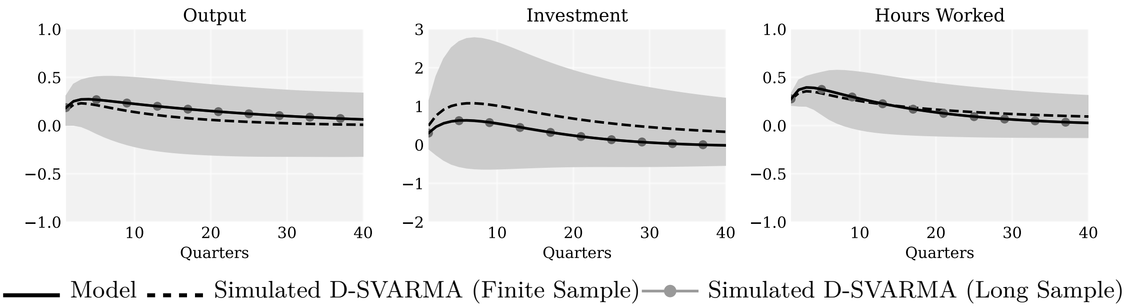
(a) Technology Shock



(b) Investment Specific Shock



(c) Labor Wedge Shock



The estimated model is a D-SVARMA(2,1). Data are generated by the calibrated RBC model. Observables are y, i and h, but k is not observable. Short sample corresponds to the average of 10,000 estimations of length 250 periods. The shaded area represents 68% confidence bands, as computed from the 10,000 simulations. Long sample corresponds to one simulation of length 1,000,000 periods.

Note that the three shocks are unlabelled in the D-SVARMA, so we order them by minimising the distance between the model structural shock and each D-SVARMA shock. Inspection of the figure suggests that the D-SVARMA allows to recover the three structural shocks: the impulse response functions share the same shape and the same properties in the model and in the D-SVARMA. In particular, the D-SVARMA is able to properly recover the short-run response to the various shocks. Closer inspection however reveals that the match is not perfect. The D-SVARMA tends to underestimate the response of output, and slightly overestimate that of hours worked. Two sources of bias are usually taken as the main culprit for this type of imperfect match: truncation bias and small sample bias. The truncation bias occurs when the state space representation of the solution is approximated by a finite VAR. This is not the case in our experiment. As explained above, the state space representation actually admits a VARMA(2,1) representation, which is precisely the unrestricted model we estimate. The truncation bias is therefore inoperative in our case. The main culprit is the small sample bias. As we increase the sample size of our simulations, the bias recedes and eventually vanishes. The grey line with bullet markers in Figure 18 corresponds to IRF obtained from the D-SVARMA estimated on a sample of 1,000,000 periods. The match is then perfect. Therefore, our dynamic identification method asymptotically correctly recovers the theoretical shocks, even in the presence of a latent variable.