Gold Rush Fever in Business Cycles
Technical Appendix

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This is a technical appendix to the paper "Gold Rush Fever in Business Cycles". In section 1, we provide some robustness checks of our results in the consumption-output VECM. In section 3, we describe in more details the analytical model of gold rush, and compute an analytical solution in the case of *i.i.d.* market shocks. In section 4, we present an analytical model with investment specific technological shock, and show that is does not mach the salient features of the date we are focusing on in the paper. In section 5, we present and estimate an monetary New–Keynesian model, and show that it does not match the data.

1 Robustness of the VECM Results

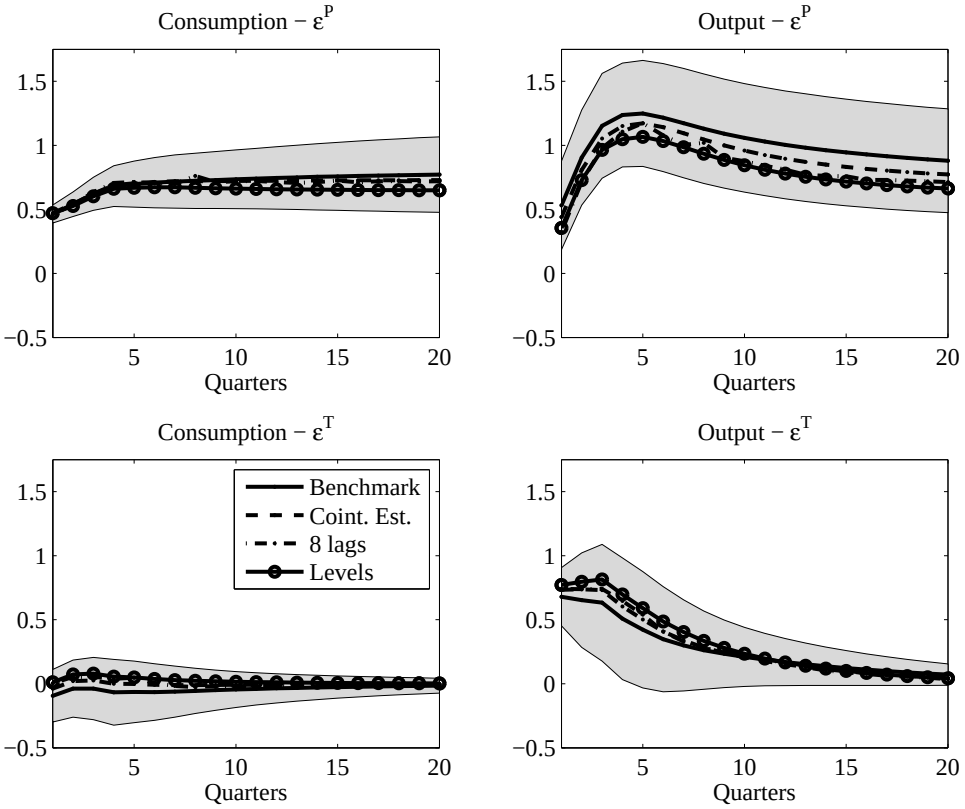
Recall that there are four properties of the data that we want to highlight: *(i)* the permanent shock to consumption (ε^P) is indeed the ε^C shock in a consumption–output VECM, *(ii)* there is virtually no dynamics in the consumption response to that shock, as it affects permanently and almost instantaneously the level of consumption, *(iii)* the temporary shock (or the output shock in the short run identification) is responsible for a significant share of output volatility at business cycle frequencies and *(iv)* hours are mainly explained by the transitory shock in the short–run. The first three facts indicate that much of the business cycle action seems to lie in investment, without any short or long run implications for consumption.

Here we investigate the robustness of these findings either against changes in the specification of the VAR — by estimating rather than imposing the cointegration relation, adding additional lags or estimating the VAR in levels — or against the data used to estimate the VAR — we considered total consumption rather than nondurables and services, output as measured by consumption plus investment only. We show that our findings are robust. We refer to the VECM of the paper as the benchmark one.

First we keep the variables (C, Y), but either estimate the cointegrating relation rather than imposing a [1;-1] cointegrating vector, use eight lags in the VECM or estimate the model in levels. As shown in Figures 1 and 2, results are strikingly robust. All impulse responses lie in the confidence band of the benchmark model.

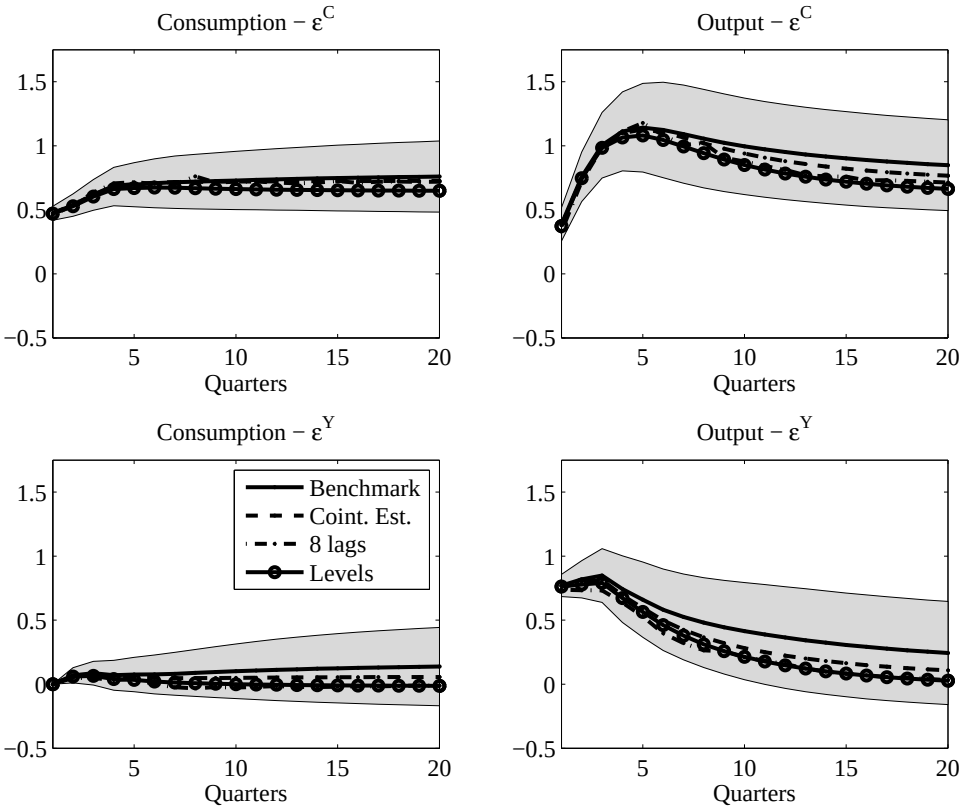
Second, we use total consumption instead of consumption of nondurables and services, or consumption plus investment instead of total output. In each case, we estimate the cointegrating relation and choose the number of lags according to likelihood ratio tests. Again, as shown on Figures 3 and 4, results are robust.

Figure 1: Robustness I (Long Run Identification)



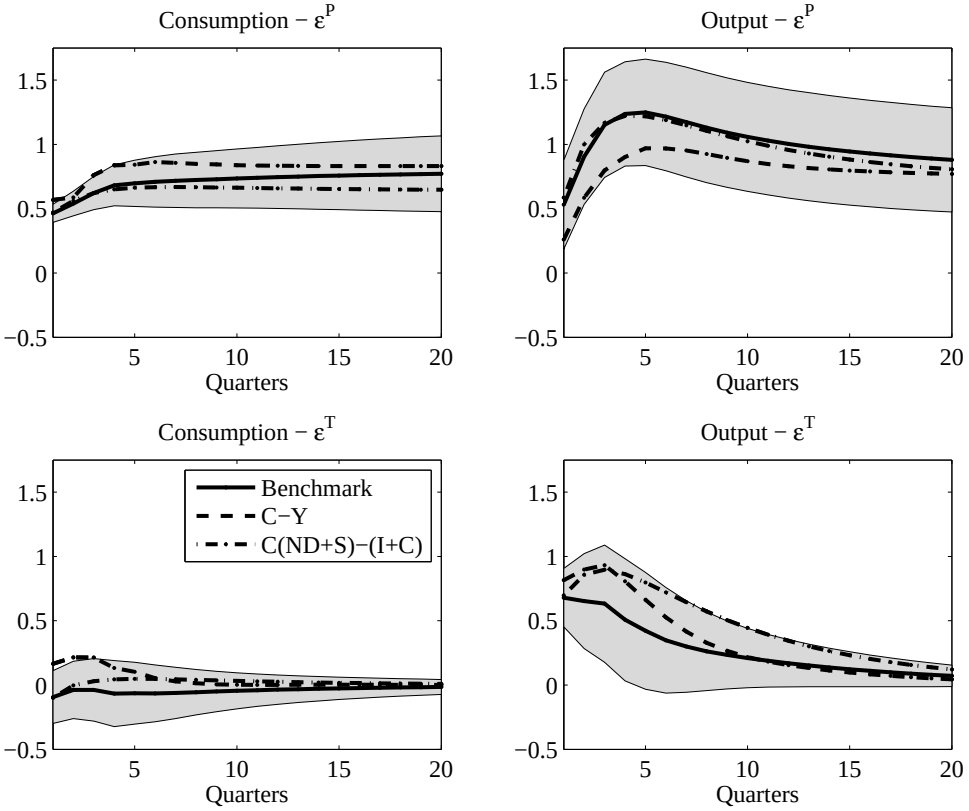
This Figure shows the response of consumption and output to temporary ε^T and permanent ε^P one percent shocks in the long run indentification, and for different specification of the model. The bold line corresponds to the benchmark VECM (C, Y) estimated with one cointegrating relation [1;-1] with 3 lags. The dashed line corresponds to a VECM in which the cointegrating relation is estimated. The dashed-dotted line corresponds to a model with eight lags of data. The line wih circles corresponds to a VAR estimated in levels (therefore with four lags of data). All estimations are done using quarterly per capita U.S. data over the period 1947Q1–2004Q4. The shaded area is the 95% confidence intervals obtained from 1000 bootstraps of the benchmark VECM.

Figure 2: Robustness I (Short Run Identification)



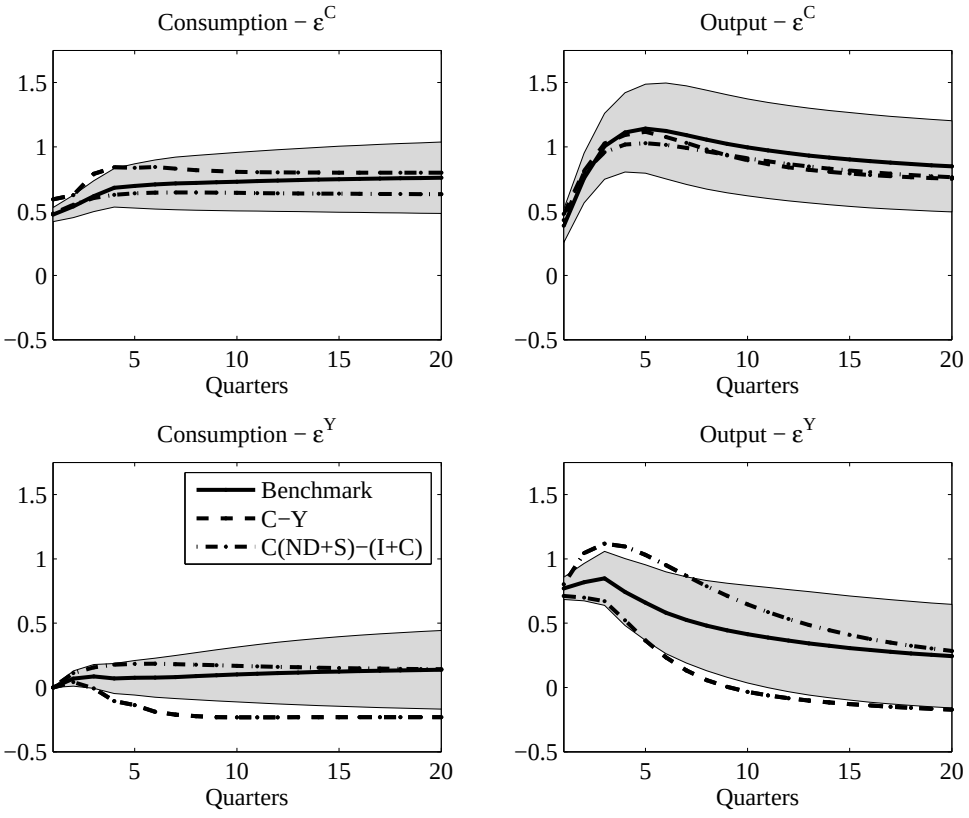
This Figure shows the response of consumption and output to consumption ε^C and output ε^Y one percent shocks in the long run indentification, and for different specification of the model. The bold line corresponds to the benchmark VECM (C, Y) estimated with one cointegrating relation [1;-1] with 3 lags. The dashed line corresponds to a VECM in which the cointegrating relation is estimated. The dashed-dotted line corresponds to a model with eight lags of data. The line wih circles corresponds to a VAR estimated in levels (therefore with four lags of data). All estimations are done using quarterly per capita U.S. data over the period 1947Q1–2004Q4. The shaded area is the 95% confidence intervals obtained from 1000 bootstraps of the benchmark VECM.

Figure 3: Robustness II (Long Run Identification)



This Figure shows the response of consumption and output to temporary ε^T and permanent ε^P one percent shock in the long run indentification, and for different ways of constructing the data. The bold line corresponds to the benchmark VECM (C, Y) estimated with one cointegrating relation [1;-1] with 3 lags, where C is the consumption of nondurable goodsand services, and Y total output. The dashed line corresponds to a VECM in which C is measured by total consumption, the cointegrating relation being estimated. The dashed-dotted line corresponds to a VECM where Y is measured by consumption plus investment instead of total output, the cointegrating relation being estimated. All estimations are done using quarterly per capita U.S. data over the period 1947Q1–2004Q4. The shaded area is the 95% confidence intervals obtained from 1000 bootstraps of the benchmark VECM.

Figure 4: Robustness II (Short Run Identification)



This Figure shows the response of consumption and output to consumption ε^C and output ε^Y one percent shocks in the long run indentification, and for different ways of constructing the data. The bold line corresponds to the benchmark VECM (C, Y) estimated with one cointegrating relation [1;-1] with 3 lags, where C is the consumption of nondurable goodsand services, and Y total output. The dashed line corresponds to a VECM in which C is measured by total consumption, the cointegrating relation being estimated. The dashed-dotted line corresponds to a VECM where Y is measured by consumption plus investment instead of total output, the cointegrating relation being estimated. All estimations are done using quarterly per capita U.S. data over the period 1947Q1–2004Q4. The shaded area is the 95% confidence intervals obtained from 1000 bootstraps of the benchmark VECM.

2 The role of investment fluctuations

In this section, we try to evaluate the role of the transitory shock in investment fluctuations. We run the regression

$$x_t = c + \sum_{k=0}^{K} \left(\alpha_k \varepsilon^P_{t-k} + \beta_k \varepsilon^T_{t-k} + \gamma_k \varepsilon^x_{t-k} \right), \tag{1}$$

where x_t denotes the log difference of investment. We consider 4 types of investment

1. Durable goods;
2. Structure;
3. Equipment and Software;
4. Residential.

This model is estimated by maximum likelihood, choosing an arbitrarily large number of lags ($K = 40$). We then compute, for each horizon k the share of the overall volatility of hours worked accounted for by ε^P, ε^T and by the investment specific shock ε^x. Results are reported in Table 1.

Table 1: Variance Decomposition

k	ε^{P}	ε^{T}	ε^{I}
(a) Durables			
1	42%	41%	17%
4	64%	30%	6%
8	66%	25%	9%
20	54%	38%	8%
40	39%	53%	8%
(b) Structures			
1	6%	51%	43%
4	38%	33%	29%
8	54%	25%	21%
20	36%	42%	22%
40	24%	52%	24%
(c) Equip. and Soft.			
1	25%	39%	36%
4	65%	29%	6%
8	77%	19%	4%
20	67%	19%	14%
40	46%	12%	42%
(d) Residential			
1	54%	17%	29%
4	52%	17%	31%
8	14%	65%	21%
20	7%	73%	20%
40	8%	51%	41%

3 An Analytical Model of Market Rushes

In this section, we make explicit the computation details in solving for our simple analytical model of market rushes. We also present an full analytical solution in the case where market shocks are *i.i.d.*.

3.1 Model

Firms : There exists a raw final good, denoted Q_t, produced by a representative firm using labor h_t and a set of intermediate goods X_{jt} with mass N_t according to a constant return to scale technology represented by the production function

$$Q_t = (\Theta_t h_t)^{\alpha} N_t^{\xi} \left(\int_0^{N_t} X_{j,t}^{\chi} \mathrm{d}j \right)^{\frac{\alpha}{\chi}}, \tag{2}$$

where Θ_t is an index of disembodied exogenous technological progress, $\alpha \in (0,1)$. $\chi \leqslant 1$ drives the elasticity of substitution between intermediate goods and ξ is a parameter that determines the long run effect of variety expansion. Since this final good will also serve to produce intermediate good, we will refer to Q_t as the gross amount of final good. Also not that the raw final good will serve as the numéraire. The representative firm is price taker on the markets.

Existing intermediate goods are produced by monopolists, who may produce more that one good. Just like in the standard expanding variety model, the production of one unit of intermediate good requires one unit of the raw final output as input. Since the final good serves as a numéraire, this leads to a situation where the price of each intermediate good is given by $P_{j,t} = \frac{1}{\chi}$. Therefore, the quantity of intermediate good j, $X_{j,t}$, produced in equilibrium is given by

$$X_{j,t} = (\chi(1-\alpha)^{\frac{1}{\alpha}} \Theta_t N_t^{\frac{\xi-1+(1-\alpha)/\chi}{\alpha}} h_t. \tag{3}$$

and the profits, $\Pi_{j,t}$, generated by intermediate firm j are given by

$$\Pi_{j,t} = \pi_0 \Theta_t N_t^{\frac{\xi-1+(1-\alpha)/\chi}{\alpha}} h_t, \tag{4}$$

where $\pi_0 = (\frac{1-\chi}{\chi})(\chi(1-\alpha))^{\frac{1}{\alpha}}$. Labor demand by the final good producer is implicitly given by

$$W_t = A\Theta N_t^{\frac{\xi+(1-\alpha)(1-\chi)/\chi}{\alpha}}, \tag{5}$$

where $A = \alpha(\chi(1-\alpha))^{\frac{(1-\alpha)}{\alpha}}$.

Value added is then given by the quantity of raw final good net of that quantity used to produce the intermediate goods. Once we substitute out for X_{jt}, and take away the amount of Q used

in the production of X_{jt}s, is given by

$$\begin{aligned} Y_t &= Q_t - \int_0^{N_t} P_{j,t} X_{j,t} dj \\ &= A\Theta_t N_t^{\frac{\xi+(1-\alpha)(1-\chi)/\chi}{\alpha}} h_t \end{aligned} \tag{6}$$

Note that π_0/A represents the share of profits in the economy, and is therefore between zero and one. This quantity will later appear a relevant parameter. Note that when $\xi = -(1-\alpha_h)(1-\chi)/\chi$, an expansion in variety exerts no long run impact. In this case, the value–added production function reduces to

$$Y_t = A\Theta_t h_t \tag{7}$$

The net amount of raw final good can serve for consumption, C_t, and startup expenditures, S_t, purposes.

$$Y_t = C_t + S_t, \tag{8}$$

Variety Dynamics : In each period, there is an exogenous probability ε_t that a potential new variety appears in the economy. In such a case, any entrepreneur who is willing to produce this potential new variety has to pay a fixed of one unit of the setup good to setup the new firm. $P_t^s S_t$ will denote the total expenditures in setup costs. A time t startup will become a functioning new firms with a product monopoly at $t+1$ with the endogenous probability ρ_t. Likewise, an existing firm/monopoly becomes obsolete at an exogenous probability μ. Therefore, the dynamics for the number of products is given by

$$N_{t+1} = (1 - \mu + \eta_t) N_t. \tag{9}$$

In the above, μN_t represents the existing products that are destroyed, while there will be $\eta_t N_t$ openings which can be filled by startups. η_t follows a random process, with unconditional mean μ. Note that η is akin to a news shock, as it is bringing some information on the future value of N_t.

The S_t startups of period t compete to secure the $\varepsilon_t N_t$ new monopoly positions. We assume that in equilibrium $S_t > \eta_t N_t$, which can later be verified as being satisfied. The $\eta_t N_t$ successful startups are drawn randomly and equiprobably among the S_t existing ones. Therefore, the probability that a startup at time t will become a functioning firm at $t+1$ is therefore given by $\rho_t = \frac{\eta_t N_t}{S_t}$.

Households : There exists an infinite number of identical households distributed over the unit interval. The preferences of the representative household are given by

$$\mathbb{E}_t \sum_{\tau=0}^{\infty} \beta^{\tau} \left[\log(C_{t+\tau}) + \psi(\overline{h} - h_{t+\tau})\right] \tag{10}$$

where $0 < \beta < 1$ is a constant discount factor, C_t denotes consumption in period t and h_t is the quantity of labor she supplies. Households choose how much to consume, supply labor, hold equity ($\mathcal{E}_t$) in existing firms, and invest in startups (S_t) maximizing 10 subject to the following budget constraint

$$C_t + P_t^{\mathcal{E}} \mathcal{E}_t + S_t = W_t h_t + \mathcal{E}_t \Pi_t + (1-\mu) P_t^{\mathcal{E}} \mathcal{E}_{t-1} + \rho_{t-1} P_t^{\mathcal{E}} S_{t-1} \tag{11}$$

where $P_t^{\mathcal{E}}$ is the beginning of period price of equity, prior to dividend payments. Dividends per equity share are assumed to be equal to period–profits Π_t. It turns out convenient rewrite the budget constraint as

$$\Omega_t + C_t = \rho_{t-1} P_t^{\mathcal{E}} \Omega_{t-1} + W_t h_t + P_t^{\mathcal{E}} \mathcal{E}_{t-1} \left(1 - \mu + \rho_{t-1}(P_{t-1}^{\mathcal{E}} - \Pi_{t-1})\right)$$

where $\Omega_t \equiv (P_t^{\mathcal{E}} - \Pi_t)\mathcal{E}_t + S_t$

The first order conditions imply

$$\begin{aligned}
\psi C_t &= W_t && (12)\\
\frac{1}{C_t} &= \lambda_t && (13)\\
\lambda_t &= \beta \rho_t \mathbb{E}_t \left[\lambda_{t+1} P_{t+1}^{\mathcal{E}}\right] && (14)\\
\beta \mathbb{E}_t \left[\lambda_{t+1} P_{t+1}^{\mathcal{E}}\right] (1 - \mu - \rho_t (P_t^{\mathcal{E}} - \Pi_t)) &= 0 && (15)
\end{aligned}$$

and the transversality condition $\lim_{t\to\infty} \beta^k \lambda_{t+k} \Omega_{t+k} = 0$

3.2 Equilibrium Allocations

The three last first order conditions can be combined to give:

$$\frac{1}{\rho_t C_t} = \beta \mathbb{E}_t \left[\frac{\Pi_{t+1}}{C_{t+1}}\right] + \beta \mathbb{E}_t \left[\frac{1-\mu}{\rho_{t+1} C_{t+1}}\right] \tag{16}$$

Using the labor demand condition (5) and the profit equation (4), the free entry condition (16) rewrites as

$$\frac{S_t}{C_t} = \beta \frac{\psi \pi_0}{A} \left(\frac{\eta_t}{1-\mu+\eta_t}\right) \mathbb{E}_t h_{t+1} + \beta \left(\frac{1-\mu}{1-\mu+\eta_t}\right) \mathbb{E}_t \left[\left(\frac{\eta_t}{\eta_{t+1}}\right) \frac{S_{t+1}}{C_{t+1}}\right], \tag{17}$$

Using the labor demand condition (5) and the resource constraint (8), we get

$$\frac{S_t}{C_t} = \psi h_t - 1, \tag{18}$$

The free entry condition can therefore we written as:

$$\left(\frac{1-\mu+\eta_t}{\eta_t}\right)(\psi h_t - 1) = \beta\frac{\psi\pi_0}{A}\mathbb{E}_t h_{t+1} + \beta\mathbb{E}_t\left[\frac{1-\mu}{\eta_{t+1}}(\psi h_{t+1} - 1)\right] \tag{19}$$

or

$$(h_t - \psi^{-1}) = \beta\frac{\psi\pi_0}{A}\delta_t\mathbb{E}_t h_{t+1} + \beta\delta_t\mathbb{E}_t\left[\left(\frac{1}{\delta_{t+1}} - 1\right)(h_{t+1} - \psi^{-1})\right]. \tag{20}$$

where $\delta_t = \eta_t/(1-\mu+\eta_t) \leqslant 1$ is a increasing function of the fraction of newly opened markets η_t.

By repeated substitution, the above equation can be written as a function of current and future values of δ only. Given the nonlinearity of equation (20), it is useful to compute a log–linear approximation around the deterministic steady–state value of employment h. The latter is given by:[1]

$$h = \frac{\psi^{-1}(1-\beta(1-\mu))}{(1-\beta\mu\frac{\pi_0}{A} - \beta(1-\mu))},$$

and the log–linear approximation takes the form

$$\widehat{h}_t = \gamma\mathbb{E}_t\widehat{h}_{t+1} + \left(\frac{h-\psi^{-1}}{h}\right)\mathbb{E}_t\left[\widehat{\delta}_t - \beta\widehat{\delta}_{t+1}\right]$$

where $\widehat{h}_t$ now represents relative deviations from the steady state and $\gamma \equiv \beta\mu(\pi_0/A) + \beta(1-\mu)$ with $\gamma \in (0,1)$. Solving forward, this can be written as

$$\widehat{h}_t = \left(\frac{h-\psi^{-1}}{h}\right)\left(\widehat{\delta}_t - \mu\beta\left(\frac{A-\pi_0}{A}\right)\mathbb{E}_t\left[\sum_{i=0}^{\infty}\gamma^i\widehat{\delta}_{t+1+i}\right]\right). \tag{21}$$

Note that, as $\gamma \in (0,1)$, the model possesses a unique determinate equilibrium path.

3.3 Deriving a Full Analytical Solution

Define $x_t = (h_t - \psi^{-1})/\delta_t$, the equation (20) rewrites

$$x_t = \Upsilon + \beta\mathbb{E}_t\left(1 - \omega\delta_{t+1}\right)x_{t+1}$$

with $\Upsilon = \beta\mu\frac{\pi_0}{\psi A}$ and $\omega = \frac{\pi_0}{A} - 1$.

[1] Note that we used the fact that $\mathbb{E}_t(\eta_t) = \mu$, which implies that $\delta = \mu$ in steady state.

Iterating forward, we obtain

$$x_t = \lim_{T\to\infty} \Upsilon \sum_{j=0}^{T} \beta^j \mathbb{E}_t \prod_{\ell=1}^{j} (1-\omega\delta_{t+\ell}) + \Upsilon \lim_{T\to\infty} \beta^T \mathbb{E}_t \prod_{\ell=1}^{T} (1-\omega\delta_{t+T}) x_{t+T}$$

Note that since $\delta_t \leqslant 1$ and $\omega < 1$, we have

$$\lim_{T\to\infty} \beta^T \mathbb{E}_t \prod_{\ell=1}^{T} (1-\omega\delta_{t+T}) x_{t+T} \leqslant \lim_{T\to\infty} \beta^T x_{t+T}$$

Furthermore, note that using the definition of Ω_t, the Euler equation (15) and the fact that $\mathcal{E}_t = 1$ in equilibrium, the transversality condition rewrites

$$\lim_{t\to\infty} \beta^k \frac{S_{t+k}}{\delta_{t+k} C_{t+k}} = 0$$

which, using (18), rewrites

$$\lim_{t\to\infty} \beta^k \frac{\psi h_{t+k} - 1}{\delta_{t+k}} = 0$$

or

$$\lim_{t\to\infty} \beta^k \frac{h_{t+k} - \psi^{-1}}{\delta_{t+k}} = \lim_{t\to\infty} \beta^k x_{t+k} = 0$$

Therefore, we have

$$x_t = \Upsilon \sum_{j=0}^{\infty} \beta^j \mathbb{E}_t \prod_{\ell=1}^{j} (1-\omega\delta_{t+\ell})$$

or

$$h_t = \psi^{-1} + \delta_t \Upsilon \sum_{j=0}^{\infty} \beta^j \mathbb{E}_t \prod_{\ell=1}^{j} (1-\omega\delta_{t+\ell})$$

Interestingly, in the case where δ_t is an *i.i.d.* process with mean δ, which is the case we consider to illustrate the Gold rush configuration in the section that follows in the paper, we have

$$h_t = \psi^{-1} + \frac{\Upsilon}{1-\beta(1-\omega\delta)} \delta_t$$

4 A model with investment specific shocks

In this section, we present a simple analytical model with investment specific shocks. The model is as close as possible to the analytical model with market shock. In the same way we have shown in the main text that our model is successful in replicating the main properties of the a consumption-output VECM, we show that this model cannot do the job. To do so, we explicitly derive the VAR representation of the model. We also derive the identified shocks of our short run and long run identifications as analytic functions of the structural ones. Regardless of the assumptions made about the stationarity of the shock, such a model cannot reproduce the salient features of the data.

4.1 Model

There exists a representative household with preferences represented by the following utility function

$$\mathbb{E}_t \sum_{\tau=0}^{\infty} \beta^{\tau} \left[\log(C_{t+\tau}) - \Psi h_{t+\tau}\right] \tag{22}$$

where $0 < \beta < 1$ is a constant discount factor, C_t denotes consumption in period t and h_t is the quantity of labor supplied by the representative household.

There exists a final good, Y, produced by means of capital, K_t and labor, h_t, according to the constant returns to scale technology represented by the Cobb–douglas function

$$Y_t = K_t^{\alpha}(\Theta_t h_t)^{1-\alpha} \text{ with } \alpha \in (0,1) \tag{23}$$

where Θ_t is an exogenous TFP shock.

Capital accumulates as

$$K_{t+1} = Q_t I_t$$

where Q_t is an investment specific shock and I_t denotes investment.

The first order conditions associated to the program of the central planner are then given by

$$\psi = (1-\alpha)\frac{Y_t}{C_t h_t} \tag{24}$$

$$\frac{1}{Q_t C_t} = \beta \mathbb{E}_t \frac{1}{C_{t+1}} \alpha \frac{Y_{t+1}}{K_{t+1}} \tag{25}$$

The Euler equation rewrites as

$$\frac{K_{t+1}}{Q_t C_t} = \beta \mathbb{E}_t \alpha \frac{Y_{t+1}}{C_{t+1}}$$

Making use of the resource constraint, we obtain

$$\frac{K_{t+1}}{Q_t C_t} = \beta \mathbb{E}_t \alpha \frac{C_{t+1} + I_{t+1}}{C_{t+1}}$$

As we have $K_{t+1} = Q_t I_t$, this rewrites

$$\frac{I_t}{C_t} = \alpha\beta \mathbb{E}_t \left[1 + \frac{I_{t+1}}{C_{t+1}} \right]$$

Iterating forward we obtain

$$\frac{I_t}{C_t} = \frac{\alpha\beta}{1 - \alpha\beta}$$

Using the resource constraint, we get

$$C_t = (1 - \alpha\beta) Y_t$$
$$I_t = \alpha\beta Y_t$$

and

$$K_{t+1} = \alpha\beta Q_t Y_t$$

Using the labor market equilibrium, we have

$$h_t = h^\star = \frac{1 - \alpha}{\psi(1 - \alpha\beta)}$$

Therefore, output is given by

$$Y_t = (Q_{t-1} I_{t-1})^\alpha (\Theta_t h^\star)^{1-\alpha}$$

which rewrites

$$Y_t = \Gamma_y (Q_{t-1} Y_{t-1})^\alpha \Theta_t^{1-\alpha}$$

with $\Gamma_y \equiv (\alpha\beta)^\alpha h^{\star 1-\alpha}$. This implies that

$$C_t = \Gamma_c (Q_{t-1} Y_{t-1})^\alpha \Theta_t^{1-\alpha}$$

with $\Gamma_c = (1 - \alpha\beta)\Gamma_y$.

Letting lowercases denote variables evaluated in logarithm, we obtain a VARMAX representation of the model solution:

$$c_t = \alpha y_{t-1} + \alpha q_{t-1} + (1 - \alpha)\theta_t$$
$$y_t = \alpha y_{t-1} + \alpha q_{t-1} + (1 - \alpha)\theta_t$$

We now derive the orthogonalized shocks ε^T and ε^P for the long run identification and ε^C and ε^Y for the short run one. Recall that in the data, we have the following properties: *(i)* the

permanent shock (ε^P) is essentially the same shock as that corresponding to a consumption shock (ε^C) , *(ii)* the response of consumption to a temporary shock is extremely close to zero at all horizons, and there are almost no dynamics in the response of consumption to a permanent shock, as it jumps almost instantaneously to its long run level, *(iii)* the temporary shock (or the output shock in the short run orthogonalization) is responsible for a significant share of output volatility at business cycle frequencies and *(iv)* the temporary shock explains much of the variance of hours at business cycle frequencies.

In order to derive the orthogonalized representations, we make different assumption regarding the processes followed by the two technological shocks.

4.2 Both Shocks Are Random Walk

We assume here that both q_t and θ_t follow a random walk. This assumption is the most natural to make, as both TFP and the relative price of investment goods are non stationary.

$$q_t = q_{t-1} + \sigma_q \varepsilon_t^q$$
$$\theta_t = \theta_{t-1} + \sigma_\theta \varepsilon_t^\theta$$

with $\mathbb{E}(\varepsilon_t^q \varepsilon_t^q) = \mathbb{E}(\varepsilon_t^\theta \varepsilon_t^\theta) = 1$ and $\mathbb{E}(\varepsilon_t^q \varepsilon_t^\theta) = 0$.

It is then straightforward to obtain the VARMA representation of the process of consumption and output growth as

$$\begin{pmatrix} 1 & -\alpha L \\ 0 & 1-\alpha L \end{pmatrix} \begin{pmatrix} \Delta c_t \\ \Delta y_t \end{pmatrix} = \begin{pmatrix} 1-\alpha & \alpha L \\ 1-\alpha & \alpha L \end{pmatrix} \begin{pmatrix} \sigma_\theta \varepsilon_t^\theta \\ \sigma_q \varepsilon_t^q \end{pmatrix}$$

Inverting the process, we obtain the following moving average representation:

$$\begin{pmatrix} \Delta c_t \\ \Delta y_t \end{pmatrix} = \begin{pmatrix} \frac{1-\alpha}{1-\alpha L} & \frac{\alpha L}{1-\alpha L} \\ \frac{1-\alpha}{1-\alpha L} & \frac{\alpha L}{1-\alpha L} \end{pmatrix} \begin{pmatrix} \sigma_\theta \varepsilon_t^\theta \\ \sigma_q \varepsilon_t^q \end{pmatrix} = C(L) \begin{pmatrix} \sigma_\theta \varepsilon_t^\theta \\ \sigma_q \varepsilon_t^q \end{pmatrix}$$

The matrix of instantaneous impact is given by:

$$C(0) = \begin{pmatrix} 1-\alpha & 0 \\ 1-\alpha & 0 \end{pmatrix}$$

When performing a short run identification, the output innovation ε^Y is obtained as the linear combination of the model shocks that has no current impact on consumption. As shown by matrix $C(0)$, the shock ε^q has no impact on consumption, and the short run orthogonalization will therefore give $\varepsilon^Y = \varepsilon^q$ and $\varepsilon_t^C = \varepsilon_t^\theta$.

Let us now turn to the long run identification. The long run matrix of impact is given by:

$$C(1) = \begin{pmatrix} 1 & \frac{\alpha}{1-\alpha} \\ 1 & \frac{\alpha}{1-\alpha} \end{pmatrix}$$

Although the two structural shocks ε^θ and ε^q are permanent, a Blanchard-Quah orthogonalization will be able to identify a permanent and a transitory shock, which obviously will be linear combinations of the two model shocks. By definition, temporary shock has no long run impact on output. From the $C(1)$ matrix, we see that a unit ε^θ shock has a long run impact of 1 on output, while a unit ε^q shock has an impact of $\alpha/(1-\alpha)$. Therefore, a simultaneous shock of α on ε^θ and $-(1-\alpha)$ on ε^q will have exactly a zeor long run impact on output, and will be the permanent shock identified by the Blanchard-Quah identification. If one also imposes unitary standard deviation, this temporary shock is :

$$\varepsilon_t^T = \frac{\alpha\sigma_q\varepsilon_t^\theta - (1-\alpha)\sigma_\theta\varepsilon_t^q}{\sqrt{\alpha^2\sigma_q^2 + (1-\alpha)^2\sigma_\theta^2}}$$

The identified permanent shock is the shock orthogonal to ε^T and with unitary variance:

$$\varepsilon_t^P = \frac{(1-\alpha)\sigma_\theta\varepsilon_t^\theta + \alpha\sigma_q\varepsilon_t^q}{\sqrt{\alpha^2\sigma_q^2 + (1-\alpha)^2\sigma_\theta^2}}$$

One can compute the correlation between ε_t^C and ε_t^P, which is given by

$$\rho(\varepsilon_t^C, \varepsilon_t^P) = \frac{(1-\alpha)\sigma_\theta}{\sqrt{\alpha^2\sigma_q^2 + (1-\alpha)^2\sigma_\theta^2}}$$

We then have

$$\rho(\varepsilon_t^C, \varepsilon_t^P) = 1 \Longleftrightarrow \sigma_q = 0$$

Hence, the only way to match the implication of the data $\varepsilon_t^C = \varepsilon_t^P$ is to assume that there is no investment specific shock. Adding permanent investment specific shocks is therefore worsening the model ability of explaining of the data.

As the reader may suspect that this result is dependant on the assumptions we made about stationarity (clearly, there are no truly structural stationary shocks in the model), we now consider cases where one of the two shock is indeed stationary.

4.3 The Investment Specific Shock Is Stationary

In this section, we will assume that θ_t is a random walk

$$\theta_t = \theta_{t-1} + \sigma_\theta\varepsilon_t^\theta$$

and that the investment specific shock is a stationary AR(1) process

$$q_t = \rho q_{t-1} + \sigma_q \varepsilon_t^q$$

with $\mathbb{E}(\varepsilon_t^q \varepsilon_t^q) = \mathbb{E}(\varepsilon_t^\theta \varepsilon_t^\theta) = 1$ and $\mathbb{E}(\varepsilon_t^q \varepsilon_t^\theta) = 0$.

The VARMA representation of the process of consumption and output growth is then given by

$$\begin{pmatrix} 1 & -\alpha L \\ 0 & 1-\alpha L \end{pmatrix} \begin{pmatrix} \Delta c_t \\ \Delta y_t \end{pmatrix} = \begin{pmatrix} 1-\alpha & \frac{\alpha L(1-L)}{1-\rho L} \\ 1-\alpha & \frac{\alpha L(1-L)}{1-\rho L} \end{pmatrix} \begin{pmatrix} \sigma_\theta \varepsilon_t^\theta \\ \sigma_q \varepsilon_t^q \end{pmatrix}$$

Inverting the process, we obtain a moving average representation of the process:

$$\begin{pmatrix} \Delta c_t \\ \Delta y_t \end{pmatrix} = \begin{pmatrix} \frac{1-\alpha}{1-\alpha L} & \frac{\alpha L(1-L)}{(1-\rho L)(1-\alpha L)} \\ \frac{1-\alpha}{1-\alpha L} & \frac{\alpha L}{(1-\rho L)(1-\alpha L)} \end{pmatrix} \begin{pmatrix} \sigma_\theta \varepsilon_t^\theta \\ \sigma_q \varepsilon_t^q \end{pmatrix} = C(L) \begin{pmatrix} \sigma_\theta \varepsilon_t^\theta \\ \sigma_q \varepsilon_t^q \end{pmatrix}$$

Since we have

$$C(0) = \begin{pmatrix} 1-\alpha & 0 \\ 1-\alpha & 0 \end{pmatrix}$$

The impact matrix of shocks has again a column of zero, and the short run identification will give $\varepsilon_t^C = \varepsilon_t^\theta$

Obviously, the Blanchard–Quah orthogonalization will imply that the permanent shock is ε_t^θ, as ε^q is now temporary. Therefore, the model replicated the property of the data regarding the equality between ε_t^P and ε_t^C, as obtained in the data. The problem with this specification of the model comes from the fact that the transitory shock, given by ε_t^q, has no instantaneous impact on output and hours, and therefore does not match some important properties of the data. The failure of the model is not only on impact. After one period, the transitory shock does move output, but also moves consumption, which is counterfactual.

4.4 The TFP Shock Is Stationary

Let us now assume that θ_t is a stationary AR(1) process

$$\theta_t = \rho \theta_{t-1} + \sigma_\theta \varepsilon_t^\theta$$

while the investment specific shock is a random walk

$$q_t = q_{t-1} + \sigma_q \varepsilon_t^q$$

with $\mathbb{E}(\varepsilon_t^q \varepsilon_t^q) = \mathbb{E}(\varepsilon_t^\theta \varepsilon_t^\theta) = 1$ and $\mathbb{E}(\varepsilon_t^q \varepsilon_t^\theta) = 0$.

The VARMA representation of the process of consumption and output growth is then given by

$$\begin{pmatrix} 1 & -\alpha L \\ 0 & 1-\alpha L \end{pmatrix} \begin{pmatrix} \Delta c_t \\ \Delta y_t \end{pmatrix} = \begin{pmatrix} \frac{(1-\alpha)(1-L)}{1-\rho L} & \alpha L \\ \frac{(1-\alpha)(1-L)}{1-\rho L} & \alpha L \end{pmatrix} \begin{pmatrix} \sigma_\theta \varepsilon_t^\theta \\ \sigma_q \varepsilon_t^q \end{pmatrix}$$

Inverting the process, we obtain the Wold decomposition of the process

$$\begin{pmatrix} \Delta c_t \\ \Delta y_t \end{pmatrix} = \begin{pmatrix} \frac{(1-\alpha)(1-L)}{(1-\alpha L)(1-\rho L)} & \frac{\alpha L}{1-\alpha L} \\ \frac{(1-\alpha)(1-L)}{(1-\alpha L)(1-\rho L)} & \frac{\alpha L}{1-\alpha L} \end{pmatrix} \begin{pmatrix} \sigma_\theta \varepsilon_t^\theta \\ \sigma_q \varepsilon_t^q \end{pmatrix} = C(L) \begin{pmatrix} \sigma_\theta \varepsilon_t^\theta \\ \sigma_q \varepsilon_t^q \end{pmatrix}$$

It is clear that in this specification the consumption shock is the technology shock, while the permanent shock is the investment specific shock. Therefore, we have $\rho(\varepsilon_t^P, \varepsilon_t^C) = 0$ which is counterfactual.

5 A monetary model

In this section, we show that a standard New–Keynesian model with technology and monetary shocks fails in reproducing the facts, as obtained from our two bivariate VECMs.

The basic set up is the New–Keynesian model with price rigidities, augmented to include various real rigidities. The production side of the economy consists of two sectors: one producing intermediate goods and the other a final good. The intermediate good is produced with capital and labor and the final good with intermediate goods. The final good is homogeneous and can be used for consumption (private and public) and investment purposes.

5.1 Final sector

The final good, Y is produced by combining intermediate goods, X_i, by perfectly competitive firms. The production function is given by

$$Y_t = \left(\int_0^1 Y_{it}^{\theta} \mathrm{d}i \right)^{\frac{1}{\theta}} \tag{26}$$

where $\theta \in (-\infty, 1)$. Profit maximization and free entry lead to the general price index

$$P_t = \left(\int_0^1 P_{it}^{\frac{\theta}{\theta-1}} \mathrm{d}i \right)^{\frac{\theta-1}{\theta}} \tag{27}$$

The final good may be used for consumption — private or public — and investment purposes.

5.2 Intermediate goods producers

Each firm i, $i \in (0, 1)$, produces an intermediate good by means of capital and labor according to a constant returns–to–scale technology, represented by the Cobb–Douglas production function

$$Y_{it} = K_{it}^{\alpha} (\Theta_t h_{it})^{1-\alpha} \text{ with } \alpha \in (0, 1) \tag{28}$$

where K_{it} and h_{it} respectively denote the physical capital and the labor input used by firm i in the production process. Θ_t is exogenous technological progress which is assumed to follow a random walk of the form

$$\log(\Theta_t) = \log(\gamma) + \log(\Theta_{t-1}) + \varepsilon_t^{\Theta}$$

Assuming that each firm i operates under perfect competition in the input markets, the firm determines its production plan by minimizing its total cost

$$\min_{\{K_{it}, h_{it}\}} P_t W_t h_{it} + P_t z_t K_{it}$$

subject to (28). This leads to the following expression for total costs:

$$P_t S_t P_{it}$$

where the real marginal cost, S, is given by $\frac{W_t^{1-\alpha} z_t^{\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}\Theta_t^{1-\alpha}}$.

Intermediate goods producers are monopolistically competitive, and therefore set prices for the good they produce. We follow Christiano et al., 20005, in assuming that firms set their prices for a stochastic number of periods. In each and every period, a firm either gets the chance to adjust its price (an event occurring with probability γ) or it does not. If it does not get the chance, then we will assume that it sets its price according to

$$P_{it} = \pi_{t-1} P_{it-1} \tag{29}$$

where π_{t-1} denotes past period inflation.

On the other hand, a firm i that sets its price optimally in period t chooses a price, $P_t^{\star}$, in order to maximize:

$$\max_{P_t^{\star}} \mathbb{E}_t \sum_{\tau=0}^{\infty} \Phi_{t+\tau}(1-\gamma)^{\tau} \left(P_t^{\star}\pi^{\star\tau} - P_{t+\tau}S_{t+\tau}\right) Y_{it+\tau}$$

subject to the total demand it faces

$$Y_{it+\tau} = \left(\frac{P_t^{\star}\Xi_{t,\tau}}{P_{t+\tau}}\right)^{\frac{1}{\theta-1}} Y_{t+\tau}$$

$\Phi_{t+\tau}$ is an appropriate discount factor derived from the household's evaluation of future relative to current consumption. $\Xi_{t,\tau}$ denotes the nominal growth component that evolves as

$$\Xi_{t,\tau} = \begin{cases} \pi_t \times \ldots \times \pi_{t+\tau-1} & \text{if } \tau \geqslant 1 \\ 1 & \text{otherwise} \end{cases}$$

This leads to the price setting equation

$$P_t^{\star} = \frac{1}{\theta} \frac{\mathbb{E}_t \sum_{\tau=0}^{\infty} (1-\gamma)^{\tau} \Phi_{t+\tau} P_{t+\tau}^{\frac{2-\theta}{1-\theta}} \Xi_{t,\tau}^{\frac{1}{\theta-1}} s_{t+\tau} y_{t+\tau}}{\mathbb{E}_t \sum_{\tau=0}^{\infty} (1-\gamma)^{\tau} \Phi_{t+\tau} \Xi_{t,\tau}^{\frac{\theta}{\theta-1}} P_{t+\tau}^{\frac{1}{\theta-1}} y_{t+\tau}} \tag{30}$$

Since the price setting scheme is independent of any firm specific characteristic, all firms that reset their prices will choose the same price.

In each period, a fraction γ of contracts ends and $(1-\gamma)$ survives. Hence, from (27) and the price mechanism, the aggregate intermediate price index writes

$$P_t = \left(\gamma P_t^{\star\,\frac{\theta}{\theta-1}} + (1-\gamma)(\pi_{t-1}P_{t-1})^{\frac{\theta}{\theta-1}}\right)^{\frac{\theta-1}{\theta}} \tag{31}$$

5.3 The Household

There exists an infinite number of households distributed over the unit interval and indexed by $j \in [0,1]$. Households have market power over the labor services they provide. The preferences of household j are given by

$$\mathbb{E}_t \sum_{\tau=0}^{\infty} \beta^{\tau} \left[\log(c_{t+\tau} - bc_{t+\tau-1}) + \nu^m \log\left(\frac{M_{t+\tau}}{P_{t+\tau}}\right) - \nu^h h_{t+\tau} \right] \tag{32}$$

where $0 < \beta < 1$ is a constant discount factor, C_t denotes consumption in period t, M_t/P_t is real balances and h_t is the quantity of labor supplied by the representative household. b is the parameter of habit persistence.

In each period, household j faces the budget constraint

$$E_t B_{t+1} Q_t + M_t + P_t(C_t + I_t) = B_t + M_{t-1} + P_t z_t K_t + P_t W_t h_t + \Omega_t + \Pi_t \tag{33}$$

where B_t is state contingent deliveries of the final good and Q_t is the corresponding price of the asset that delivers these goods. M_t is end of period t money holdings. P_t, the nominal price of goods. C_t and I_t are consumption and investment expenditure respectively; K_t is the amount of physical capital owned by the household and leased to the firms at the real rental rate z_t. W_t is the nominal wage. Ω_t is a nominal lump-sum transfer received from the monetary authority and Π_t denotes the profits distributed to the household by the firms.

Capital accumulates according to the law of motion

$$K_{t+1} = \left[1 - \Phi\left(\frac{I_t}{I_{t-1}}\right) \right] I_t + (1-\delta) K_t \tag{34}$$

where $\delta \in [0,1]$ denotes the rate of depreciation. $\Phi(\cdot)$ accounts for the existence of investment adjustment costs and satisfies $\Phi(\gamma) = \Phi'(\gamma) = 0$ and and $\gamma S''(\Gamma) = \varphi > 0$.

5.4 The monetary authorities

Our specification of monetary policy involves an exogenous money supply rule, where money evolves according to

$$M_t = \exp(g_t) M_{t-1} \tag{35}$$

The gross growth rate of the money supply, g_t, is assumed to follow an exogenous AR(1) stochastic process of the form

$$g_t = \rho_g g_{t-1} + (1-\rho_g)\overline{g} + \varepsilon_t^g$$

where $|\rho_g| < 1$ and $\varepsilon_t^g \rightsquigarrow \mathrm{N}(0, \sigma_g^2)$.

5.5 Parametrization

The parameters are reported in Table 2. All of them are standardly used in the literature. Three

Table 2: Parametrization: Monetary Model

Preferences		
Discount factor	β	0.9926
Technology		
Capital elasticity of intermediate output	α	0.281
Parameter of markup	θ	0.850
Depreciation rate	δ	0.025
Probability of price resetting	γ	0.250
Shocks and policy parameters		
Persistence of money growth	ρ_g	0.500
Volatility of money shock	σ_g	0.007
Steady state money supply growth (gross)	g	0.0012

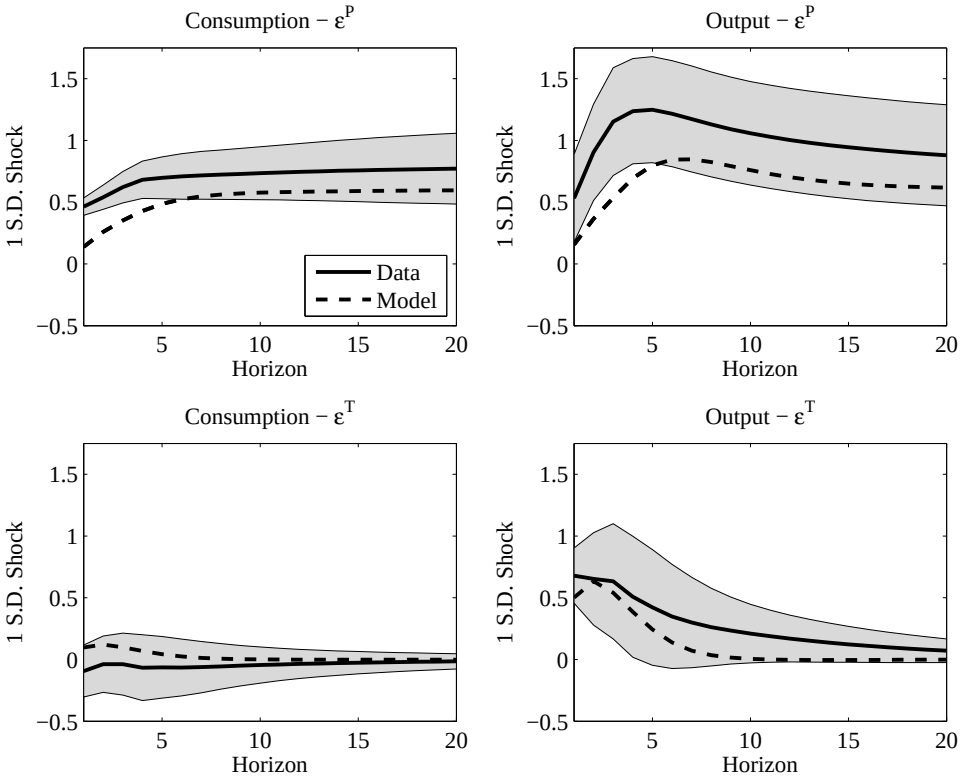
remaining parameters are estimated by a simulated method of moments. They are chosen in order to match the impulse response functions of consumption and output to a shock on the permanent (resp. on the transitory) component in the VECM. This led to the parameters reported in table (3). Note that the estimated model is substantially at odds with the data

Table 3: Estimated Coefficients

σ_Θ	b	φ	J–stat
0.0084	1.4091	0.5002	250.75
(0.0012)	(0.0655)	(0.0760)	[0.00]

since the J–stat is close to 250. In order to help visualize where this model fails, in Figure 5 we report the impulse responses associated with using the long run restrictions to orthogonalize the VECM residuals. As can be seen, the response of consumption and output to an output shock in the estimated model do not line up with those obtained in the data, especiallly as far as the permanent component is concerned. Furthermore, the examination of Figure 6 reveals that the transitory shock as identified by the long run scheme and the output shocks as identified by the impact scheme differ. For this reason, we conclude that such a model offers a poor explanation to the properties of consumption and output. Furthermore, the variance decomposition in the theoretical model indicates that the monetary shock accounts for all the short–run volatility of output.

Figure 5: Impulse Response Functions VAR versus Monetary Model (LR orthogonalization)



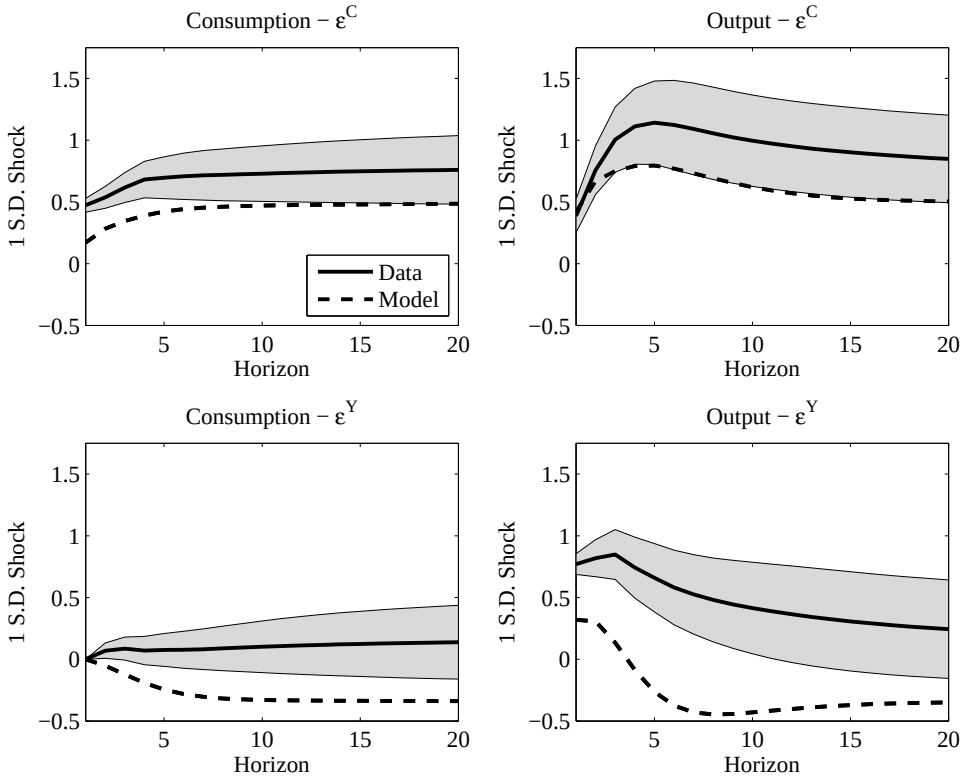
This figure compares the responses of consumption and output to permanent and transitory shocks (long run orthogonalization scheme), as estimated from the data (continuous line) and from model simulated data (dashed line). More precisely, the dashed line is the average over the 20 replications of the model used during estimation, VECM estimation and orthogonalization. The shaded area represents the 95% confidence intervals obtained from 1000 bootstraps of the VECM estimated with actual data.

Table 4: Variance decomposition

Horizon	Output		Consumption		Hours	
	ε^{Θ}	ε^{g}	ε^{Θ}	ε^{g}	ε^{Θ}	ε^{g}
1	1%	99%	40%	60%	60%	40%
4	21%	79%	73%	27%	33%	67%
8	68%	32%	92%	8%	38%	62%
20	88%	12%	98%	2%	41%	59%
∞	100%	0%	100%	0%	42%	58%

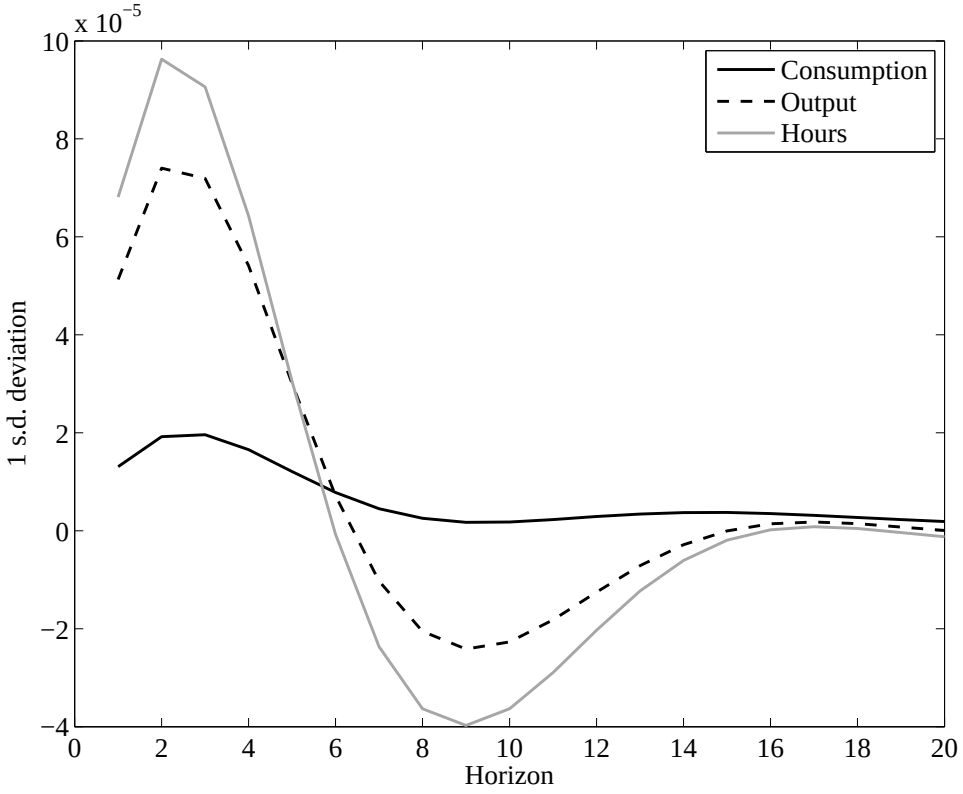
This table reports the forecast error variance decomposition of consumption, output and hours worked when the estimated model is used as the forecasting model.

Figure 6: Impulse Response Functions VAR versus Monetary Model (SR orthogonalization)



This figure compares the responses of consumption and output to consumption and output shocks (short run orthogonalization scheme), as estimated from the data (continuous line) and from model simulated data (dashed line). More precisely, the dashed line is the average over the 20 replications of the model used during estimation, VECM estimation and orthogonalization. The shaded area represents the 95% confidence intervals obtained from 1000 bootstraps of the VECM estimated with actual data.

Figure 7: Model Response Functions to a Monetary Shock



This figure displays the responses of consumption, investment, hours worked and output to a monetaryy innovation of one standard-deviation, as computed from the estimated model.