Solving Linear RE Models: A Note on Klein's Approach

1 The Problem

1.1 Setting up the problem

Our goal is to solve for the following problem

$$N_y Y_t = N_x X_t + N_z Z_t \tag{1}$$

$$M_{x0} E_t X_{t+1} + M_{y0} E_t Y_{t+1} + M_{z0} E_t Z_{t+1} = M_{x1} X_t + M_{y1} Y_t + M_{z1} Z_t \tag{2}$$

$$Z_t = \Phi Z_{t-1} + \Psi \varepsilon_t \tag{3}$$

where Y_t is $(n_y \times 1)$, X_t is $(n_x \times 1)$ and Z_t is $(n_z \times 1)$. Y_t is the set of variables of interest which are actually defined by measurement equations. X_t is the $(n_x \times 1)$ vector of state and co–state variables which can be partitioned in two parts as

$$X_t = \begin{pmatrix} X_t^b \\ X_t^f \end{pmatrix}$$

The first part pertains to the n_b predetermined variables X_t^b. The second part collects the n_f jump variables X_t^f. We therefore have $n_x = n_b + n_f$. Z_t is the $(n_z \times 1)$ vector of forcing variables. Therefore the matrices have the following dimensions:

$$\begin{array}{ccc} N_y & N_x & N_z \\ (n_y \times n_y) & (n_y \times n_x) & (n_y \times n_z) \\ \\ M_{x0} & M_{y0} & M_{z0} \\ (n_x \times n_x) & (n_x \times n_y) & (n_x \times n_z) \\ \\ M_{x1} & M_{y1} & M_{z1} \\ (n_x \times n_x) & (n_x \times n_y) & (n_x \times n_z) \\ \\ \Phi & \Psi & \\ (n_z \times n_z) & (n_z \times n_e) & \end{array}$$

N_y is assumed to be invertible. Φ has all eigenvalues lying within the unit circle and $\varepsilon \rightsquigarrow \mathscr{N}(0, \Sigma)$.

1.2 Transforming the problem

It is useful to slightly transform the problem to get rid off some variables — namely the variables of interest Y_t which can easily be eliminated leaving the dynamic properties of the model intact.

From the measurement equation, since N_y is invertible, we get

$$Y_t = N_y^{-1} N_x X_t + N_y^{-1} N_z Z_t$$

Furthermore, from the definition of the forcing variables Z_t, we have $E_t Z_{t+1} = \Phi Z_t$. Hence the dynamics reduce to

$$A E_t X_{t+1} = B X_t + C Z_t$$

with

$$\begin{aligned} A =& M_{x0} + M_{y0} N_y^{-1} N_x \\ B =& M_{x1} + M_{y1} N_y^{-1} N_x \\ C =& M_{z1} + M_{y1} N_y^{-1} N_z - (M_{z0} + M_{y0} N_y^{-1} N_z)\Phi \end{aligned}$$

2 Solving the System

Let us first recover the generalized Schur decomposition of the pencil (A,B). We therefore get the unitary $n_x \times n_x$ matrices of complex numbers Q and Z such that $S = QAZ$ and $T = QBZ$ are upper triangular, and $QQ' = ZZ' = I$.

Then the dynamics equation can be rewritten as

$$AZZ' E_t X_{t+1} = BZZ' X_t + C Z_t$$

Let us define $\omega_t = Z' X_t$ to get

$$AZ E_t \omega_{t+1} = BZ \omega_t + C Z_t$$

and premultiply by Q

$$QAZ E_t \omega_{t+1} = QBZ \omega_t + C Z_t$$

which rewrites

$$S E_t \omega_{t+1} = T \omega_t + R Z_t$$

with $R = QC$.

T_{ii}/S_{ii} are the Generalized eigenvalues of the system. Hereafter we will assume that the Schur decomposition is cooked such that eigenvalues are sorted in ascending order. We have n_s stable eigenvalues, modulus below unity, to which we associate the vector ω_t^s. We then have n_f unstable eigenvalues, modulus greater than unity, to which we associate the vector ω_t^f.

Proposition 1 *Blanchard and Kahn condition If $n_b = n_s$ (and $n_u = n_f$) then the system admits a unique saddle path.*

We then partition the system according to the partition of eigenvalues

$$Z = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} S = \begin{pmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{pmatrix} T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix}$$

where Z_{11} is $(n_s \times n_s)$, Z_{12} is $(n_s \times n_f)$, Z_{21} is $(n_f \times n_s)$, and Z_{22} is $(n_f \times n_f)$. The same applies to T and S

The system then rewrites

$$\begin{pmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{pmatrix} \begin{pmatrix} E_t\omega^b_{t+1} \\ E_t\omega^f_{t+1} \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix} \begin{pmatrix} \omega^b_t \\ \omega^f_t \end{pmatrix} + \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} Z_t$$

3 Solving for the feedforward part

Let's focus on the the unstable part of the system

$$S_{22}E_t\omega^f_{t+1} = T_{22}\omega^f_t + R_2 Z_t$$

Since T_{22} is invertible, this may be reexpressed as

$$\omega^f_t = T_{22}^{-1}S_{22}E_t\omega^f_{t+1} - T_{22}^{-1}R_2 Z_t$$

Since we are dealing with the unstable part of the system, the diagonal elements of $(T_{22}^{-1}S_{22})$ are less than 1 in modulus. It is then possible to develop the preceding equation forward to get

$$\omega^f_t = \lim_{k\longrightarrow\infty}(T_{22}^{-1}S_{22})^k E_t\omega^f_{t+k} - \sum_{k=0}^{\infty}(T_{22}^{-1}S_{22})^k T_{22}^{-1}Q_2\Phi^k z_t$$

Since we are focusing on bounded solutions — *i.e.* solution that satisfy $E_t\omega^f_{t+k} < \infty$, and since $T_{22}^{-1}S_{22}$ has modulus less than unity, it has to be the case that

$$\lim_{k\longrightarrow\infty}(T_{22}^{-1}S_{22})^k E_t\omega^f_{t+k} = 0$$

Therefore

$$\omega^f_t = -\sum_{k=0}^{\infty}(T_{22}^{-1}S_{22})^k T_{22}^{-1}Q_2\Phi^k z_t = \Gamma z_t$$

Furthermore, we have

$$\text{vec}(A+B) = \text{vec}(A) + \text{vec}(B)$$
$$\text{vec}(ABC) = (C' \otimes A)\text{vec}(B)$$
$$(AB \otimes CD) = (A \otimes C)(B \otimes D)$$

and

$$S = \sum_{k=0}^{\infty} A^k B C^k = B + A\,S\,C$$

Therefore,

$$\Gamma = -T_{22}^{-1} Q_2 + (T_{22}^{-1} S_{22}) \Gamma \Phi$$

Then

$$\begin{aligned}
\mathrm{vec}(\Gamma) &= -\mathrm{vec}(T_{22}^{-1} Q_2) + \mathrm{vec}((T_{22}^{-1} S_{22}) \Gamma \Phi) \\
\mathrm{vec}(\Gamma) &= -(I \otimes T_{22}^{-1}) \mathrm{vec}(Q_2) + (\Phi' \otimes (T_{22}^{-1} S_{22})) \mathrm{vec}(\Gamma) \\
\mathrm{vec}(\Gamma) &= -(I \otimes T_{22}^{-1}) \mathrm{vec}(Q_2) + (\Phi' \otimes S_{22})(I \otimes (T_{22}^{-1}) \mathrm{vec}(\Gamma) \\
(I \otimes T_{22}) \mathrm{vec}(\Gamma) &= -\mathrm{vec}(Q_2) + (\Phi' \otimes S_{22}) \mathrm{vec}(\Gamma)
\end{aligned}$$

Γ is then given by

$$\mathrm{vec}(\Gamma) = (\Phi' \otimes S_{22} - I \otimes T_{22})^{-1} \mathrm{vec}(Q_2)$$

We therefore know that

$$\omega_t^f = \Gamma Z_t$$

Recalling that $\omega_t^f = [Z'_{12} Z'_{22}] X_t$, we have

$$Z'_{12} X_t^b + Z'_{22} X_t^f = \Gamma Z_t$$

Since the equation is linear in X_t^b and Z_t, a solution for X_t^f is of the kind

$$X_t^f = \alpha X_t^b + \beta Z^t$$

Plugging in the preceding expression

$$Z'_{12} X_t^b + Z'_{22} (\alpha X_t^b + \beta Z^t) = \Gamma Z_t$$

Identifying term by term, we have to solve

$$Z'_{12} + Z'_{22} \alpha = 0 \quad (4)$$

$$Z'_{22} \beta = \Gamma \quad (5)$$

$$(6)$$

for α and β.

Let us focus on α. Since $Z'Z = I$, it has to be the case that

$$Z'_{12} Z_{11} + Z'_{22} Z_{21} = 0$$

As Z_{11} is invertible, this rewrites

$$Z'_{12} + Z'_{22}Z_{21}Z_{11}^{-1} = 0$$

It is then clear that $\alpha = Z_{21}Z_{11}^{-1}$ is a solution to equation 4.

Let us now focus on β. Let us first define $\widetilde{\beta}$ such that $\beta = \widetilde{\beta}M$ (5) then rewrites

$$Z'_{22}\widetilde{\beta}\Gamma = \Gamma$$

Once again, making use of $Z'Z = I$, we have

$$\begin{aligned} Z'_{12}Z_{11} + Z'_{22}Z_{21} &= 0 \\ Z'_{12}Z_{12} + Z'_{22}Z_{22} &= I \end{aligned}$$

From the first equation and as Z_{11} invertible, we have

$$Z'_{12} = -Z'_{22}Z_{21}Z_{11}^{-1}$$

Plugging this result in the second equation, we get

$$Z'_{22}(Z_{22} - Z_{21}Z_{11}^{-1}Z_{12}) = I$$

Finally, post–multiplying by Γ

$$Z'_{22}(Z_{22} - Z_{21}Z_{11}^{-1}Z_{12})\Gamma = \Gamma$$

It is then clear that $\beta = (Z_{22} - Z_{21}Z_{11}^{-1}Z_{12})\Gamma$ is a solution to (5).

Therefore, we are done with the computation of the forward part of the solution.

$$X_t^f = Z_{21}Z_{11}^{-1}X_t^b + (Z_{22} - Z_{21}Z_{11}^{-1}Z_{12})\Gamma Z_t$$

or

$$X_t^f = F_x X_t^b + F_z Z_t$$

3.1 The backward part of the solution

Let us focus on the upper part of the transformed system

$$S_{11}E_t\omega_{t+1}^b + S_{12}E_t\omega_{t+1}^f = T_{12}\omega_t^f + T_{11}\omega_t^f + R_1 Z_t$$

Since, S_{11} is invertible, this rewrites as or

$$E_t\omega_{t+1}^b = S_{11}^{-1}T_{11}\omega_t^b + S_{11}^{-1}T_{12}\omega_t^f - S_{11}^{-1}S_{12}E_t\omega_{t+1}^f + S_{11}^{-1}R_1 Z_t$$

Recall that

$$\omega_t = Z' X_t \Longleftarrow \omega_t^b = Z'_{11} X_t^b + Z'_{21} X_t^f$$

such that, making use of the solution for X_t^f

$$\omega_t^b = (Z'_{11} + Z'_{21} Z_{21} Z_{11}^{-1}) X_t^b + Z'_{21}(Z_{22} - Z_{21} Z_{11}^{-1} Z_{12}) \Gamma Z_t$$

Since, Z is a unitary matrix, we have

$$Z'_{11} Z_{11} + Z'_{21} Z_{21} = I \Longrightarrow Z'_{11} + Z'_{21} Z_{21} Z_{11}^{-1} = Z_{11}^{-1}$$

Likewise,

$$Z'_{21} Z_{22} + Z'_{11} Z_{12} = 0 \Longrightarrow Z'_{21} Z_{22} = -Z'_{11} Z_{12}$$

Therefore:

$$\begin{aligned} Z'_{21}(Z_{22} - Z_{21} Z_{11}^{-1} Z_{12}) &= Z'_{21} Z_{22} - Z'_{21} Z_{21} Z_{11}^{-1} Z_{12} \\ &= -(Z'_{11} + Z'_{21} Z_{21} Z_{11}^{-1}) Z_{12} \\ &= -Z_{11}^{-1} Z_{12} \end{aligned}$$

and $\omega_t^b = Z_{11}^{-1} X_t^b - Z_{11}^{-1} Z_{12} \Gamma Z_t$

Let us now use the fact that X_t^b corresponds to the predetermined variables, implying that

$$X_{t+1}^b - E_t X_{t+1}^b = 0$$

which yields

$$Z_{11}(\omega_{t+1}^b - E_t \omega_{t+1}^b) + Z_{12}(\omega_{t+1}^f - E_t \omega_{t+1}^f) = 0$$

Since Z_{11} is invertible

$$\omega_{t+1}^b = E_t \omega_{t+1}^b - Z_{11}^{-1} Z_{12}(\omega_{t+1}^f - E_t \omega_{t+1}^b)$$

or

$$\omega_{t+1}^b = E_t \omega_{t+1}^b - Z_{11}^{-1} Z_{12} \Gamma \varepsilon_{t+1}$$

Making use of the dynamics of ω_t^b, we get

$$\omega_{t+1}^b = S_{11}^{-1} T_{11} \omega_t^b + S_{11}^{-1}(T_{12}\Gamma - S_{12}\Gamma\Phi + R_1) Z_t - Z_{11}^{-1} Z_{12} \Gamma \varepsilon_{t+1}$$

Since $\omega_t^b = Z_{11}^{-1} X_t^b - Z_{11}^{-1} Z_{12} \Gamma Z_t$, we have

$$Z_{11}^{-1} X_{t+1}^b - Z_{11}^{-1} Z_{12} \Gamma Z_{t+1} = S_{11}^{-1} T_{11}(Z_{11}^{-1} X_t^b - Z_{11}^{-1} Z_{12} \Gamma Z_t) + S_{11}^{-1}(T_{12}\Gamma - S_{12}\Gamma\Phi + R_1) Z_t - Z_{11}^{-1} Z_{12} \Gamma \varepsilon_{t+1}$$

As Z_{11} is invertible, this rewrites

$$X_{t+1}^b - Z_{12} \Gamma Z_{t+1} = Z_{11} S_{11}^{-1} T_{11}(Z_{11}^{-1} X_t^b - Z_{11}^{-1} Z_{12} \Gamma Z_t) + Z_{11} S_{11}^{-1}(T_{12}\Gamma - S_{12}\Gamma\Phi + R_1) Z_t - Z_{12} \Gamma \varepsilon_{t+1}$$

As $Z_{t+1} = \Phi Z_t + \varepsilon_{t+1}$, we have

$$X^b_{t+1} = Z_{11}S_{11}^{-1}T_{11}Z_{11}^{-1}X^b_t + [Z_{11}S_{11}^{-1}(T_{12}\Gamma - S_{11}^{-1}S_{12}\Gamma\Phi - T_{11}Z_{11}^{-1}Z_{12}\Gamma + R_1) + Z_{12}\Gamma\Phi]Z_t$$

Therefore, the dynamics of predetermined variables is given by

$$X^b_{t+1} = M_x X^b_t + M_z Z_t$$

with

$$\begin{aligned} M_x &= Z_{11}S_{11}^{-1}T_{11}Z_{11}^{-1} \\ M_x &= Z_{11}S_{11}^{-1}(T_{12}\Gamma - S_{11}^{-1}S_{12}\Gamma\Phi - T_{11}Z_{11}^{-1}Z_{12}\Gamma + R_1) + Z_{12}\Gamma\Phi \end{aligned}$$

3.2 Variables of interest

Recall we have

$$N_y Y_t = N_x X_t + P_z Z_t = N_{x1} X^b_t + N_{x2} X^f_t + N_z Z_t$$

Using the solution for X^f_t

$$N_y Y_t = N_x X_t + N_z Z_t = N_{x1} X^b_t + N_{x2}(F_x X^b_t + F_z Z_t) + N_z Z_t$$

so that

$$Y_t = N_y^{-1}(N_x 1 + N_{x2}F_x)X^b_t + N_y^{-1}(N_z + N_{x2}F_z)Z_t$$

or

$$Y_t = P_x X^b_t + P_z Z_t$$

Finally, the solution of the whole dynamic system is given by the following stat–space form

$$\left\{ \begin{array}{rcl} X^b_{t+1} &=& M_x X^b_t + M_z Z_t \\ Z_{t+1} &=& \Phi Z_t + \varepsilon_{t+1} \\ X^f_t &=& F_x X^b_t + F_z Z_t \\ Y_t &=& P_x X^b_t + P_z Z_t \end{array} \right.$$