Chapter 1

Expectations and Economic Dynamics

Expectations lie at the core of economic dynamics as they usually determine, not only the behavior of the agents, but also the main properties of the economy under study. Although having been soon recognized, the question of expectations has been neglected for a while, as this is a pretty difficult issue to deal with. In this course, we will mainly be interested by *"rational expectations"*

1.1 The rational expectations hypothesis

The term "rational expectations" is most closely associated with Nobel Laureate Robert Lucas of the University of Chicago, but the question of rationality of expectations came into the place before Lucas investigated the issue (see Muth [1960] or Muth [1961]). The most basic interpretation of rational expectations is usually summarized by the following statement:

> *Individuals do not make systematic errors in forming their expectations; expectations errors are corrected immediately, so that — on average — expectations are correct.*

But rational expectation is a bit more subtil concept that may be defined in 3 ways.

Definition 1 (Broad definition) *Rational expectations are such that individuals formulate their expectations in an optimal way, which is actually com-*

parable to economic optimization.

This definition actually states that individuals collect information about the economic environment and use it in an optimal way to specify their expectations. For example, assume that an individual wants to make forecasts on an asset price, she needs to know the series of future dividends and therefore needs to make predictions about these dividends. She will then collect all available information about the environment of the firm (expected demand, investments, state of the market...) and use this information in an optimal way to make expectations. But two key issues emerge then: *(i)* the cost of collecting information and *(ii)* the definition of the objective function. Hence, despite its general formulation, this definition remains weakly operative. Therefore, a second definition was proposed in the literature.

Definition 2 (mid–definition) *Agents do not waste any available piece of information and use it to make the best possible fit of the real world.*

This definition has the great advantage of avoiding to deal with the problem of the cost of collecting information — we only need to know that agents do not waste information — but it remains weakly operative in the sense it is not mathematically specified. Hence, the following weak definition is most commonly used.

Definition 3 (weak definition) *Agents formulate expectations in such a way that their subjective probability distribution of economic variables (conditional on the available information) coincides with the objective probability distribution of the same variable (the state of Nature) in an equilibrium:*

$$x_t^e = E(x_t|\Omega)$$

where Ω *denote the information set*

When the model satisfies a *markovian* property, Ω essentially consists of past realizations of the stochastic variables from t=0 on. For instance, if we go back to our individual who wants to predict the price of an asset in period t, Ω will essentially consist of all past realizations of this asset price: $\Omega = \{p_{t-i}; i = 1 \ldots t\}$. Beyond, this definition assumes that agents know the model and the probability distributions of the shocks that hit the economy — that is what is needed to compute all the moments (average, standard deviations, covariances ...) which are needed to compute expectations. In other words, and this is precisely what makes rational expectations so attractive:

> **Expectations should be consistent with the model**
> $\Longrightarrow$ **Solving the model is finding an expectation function**.

Notation: Hereafter, we will essentially deal with markovian models, and will work with the following notation:

$$E_{t-i}(x_t) = E(x_t|\Omega_{t-i})$$

where $\Omega_{t-i} = \{x_k; k = 0 \ldots t-i\}$.

The weak definition of rational expectations satisfies two vary important properties.

Proposition 1 *Rational Expectations do not exhibit any bias: Let* $\widehat{x}_t = x_t - x_t^e$ *denote the expectation error:*

$$E_{t-1}(\widehat{x}_t) = 0$$

which essentially corresponds to the fact that individuals do not make systematic errors in forming their expectations.

Proposition 2 *Expectation errors do not exhibit any serial correlation:*

$$\begin{aligned} Cov_{t-1}(\widehat{x}_t, \widehat{x}_{t-1}) &= E_{t-1}(\widehat{x}_t \widehat{x}_{t-1}) - E_{t-1}(\widehat{x}_t)E_{t-1}(\widehat{x}_{t-1}) \\ &= E_{t-1}(\widehat{x}_t)\widehat{x}_{t-1} - E_{t-1}(\widehat{x}_t)\widehat{x}_{t-1} \\ &= 0 \end{aligned}$$

Example 1 *Let's consider the following AR(2) process*

$$x_t = \varphi_1 x_{t-1} + \varphi_2 x_{t-2} + \varepsilon_t$$

such that the roots lies outside the unit circle and ε_t *is the innovation of the process.*

1. *Let's now specify* $\Omega = \{x_k; k = 0, \ldots, t-1\}$, *then*

$$\begin{aligned} E(x_t|\Omega) &= E(\varphi_1 x_{t-1} + \varphi_2 x_{t-2} + \varepsilon_t|\Omega) \\ &= E(\varphi_1 x_{t-1}|\Omega) + E(\varphi_2 x_{t-2}|\Omega) + E(\varepsilon_t|\Omega) \end{aligned}$$

Note that by construction, we have $x_{t-1} \in \Omega$ *and* $x_{t-2} \in \Omega$, *therefore,* $E(x_{t-1}|\Omega) = x_{t-1}$ *and* $E(x_{t-2}|\Omega) = x_{t-2}$. *Since,* ε_t *is an innovation, it is orthogonal to any past realization of the process,* $\varepsilon_t \perp \Omega$ *such that* $E(\varepsilon_t|\Omega) = 0$. *Hence*

$$E(x_t|\Omega) = \varphi_1 x_{t-1} + \varphi_2 x_{t-2}$$

2. *Let's now specify* $\Omega = \{x_k; k = 0, \ldots, t-2\}$, *then*

$$\begin{aligned} E(x_t|\Omega) &= E(\varphi_1 x_{t-1} + \varphi_2 x_{t-2} + \varepsilon_t|\Omega) \\ &= E(\varphi_1 x_{t-1}|\Omega) + E(\varphi_2 x_{t-2}|\Omega) + E(\varepsilon_t|\Omega) \end{aligned}$$

Note that by construction, we have $x_{t-2} \in \Omega$, *such that as before* $E(x_{t-2}|\Omega) = x_{t-2}$. *Further, we still have* $\varepsilon_t \perp \Omega$ *such that* $E(\varepsilon_t|\Omega) = 0$. *But now* $x_{t-1} \notin \Omega$ *such that*

$$E(x_t|\Omega) = \varphi_1 E(x_{t-1}|\Omega) + \varphi_2 x_{t-2}$$

and we shall compute $E(x_{t-1}|\Omega)$:

$$\begin{aligned} E(x_{t-1}|\Omega) &= E(\varphi_1 x_{t-2} + \varphi_2 x_{t-3} + \varepsilon_{t-1}|\Omega) \\ &= E(\varphi_1 x_{t-2}|\Omega) + E(\varphi_2 x_{t-3}|\Omega) + E(\varepsilon_{t-1}|\Omega) \end{aligned}$$

Note that $x_{t-2} \in \Omega$, $x_{t-3} \in \Omega$ *and* $\varepsilon_{t-1} \perp \Omega$, *such that*

$$E(x_{t-1}|\Omega) = \varphi_1 x_{t-2} + \varphi_2 x_{t-3}$$

Hence

$$E(x_t|\Omega) = (\varphi_1^2 + \varphi_2) x_{t-2} + \varphi_2 x_{t-3}$$

This example illustrates the so called law of iterated projection.

Proposition 3 (Law of Iterated Projection) *Let's consider two information sets* Ω_t *and* Ω_{t-1}, *such that* $\Omega_t \supset \Omega_{t-1}$, *then*

$$E(x_t|\Omega_{t-1}) = E(E(x_t|\Omega_t)|\Omega_{t-1})$$

Beyond, the example reveals a very important property of rational expectations: **a rational expectation model is not a model in which the individual knows everything**. Everything depends on the information structure. Let's consider some simple examples.

Example 2 (signal extraction) *In this example, we will deal with a situation where the agents know the model but do not perfectly observe the shocks they face. Information is therefore incomplete because the agents do not know perfectly the distribution of the "true" shocks.*

Assume that a firm wants to predict the demand, d, it will be addressed, but only observes a random variable x that is related to d as

$$x = d + \eta \tag{1.1}$$

where $E(d\eta) = 0$, $E(d^2) = \sigma_d < \infty$, $E(\eta^2) = \sigma_\eta < \infty$, $E(d) = \delta$, and $E(\eta) = 0$. This assumption amounts to state that x differs from d by a measurement error, η. Note that in this example, we assume that there is a noisy information, but the firm still knows the overall structure of the model (namely it knows 1.1). The problem of the firm is then to formulate an expectation for d only observing x: $\Omega = \{1, x\}$. In this case, the problem of the entrepreneur is to determine $E(d|\Omega)$. Since the entrepreneur knows the linear structure of the model, it can guess that

$$E(d|\Omega) = \alpha_0 + \alpha_1 x$$

From proposition 1, we know that the expectation error exhibits no bias so that

$$E(d - E(d|\Omega)|\Omega) = 0$$

which amounts to

$$E(d - \alpha_0 - \alpha_1 x|\Omega) = 0$$

or

$$\begin{cases} E(d - \alpha_0 - \alpha_1 x|1) = 0 \\ E(d - \alpha_0 - \alpha_1 x|x) = 0 \end{cases}$$

These are the two normal equation associated with an OLS estimate, hence we have

$$\alpha_1 = \frac{Cov(x,d)}{V(x)} = \frac{Cov(d+\eta,d)}{V(d+\eta)} = \frac{\sigma_d^2}{\sigma_d^2 + \sigma_\eta^2}$$

and

$$\alpha_0 = \frac{\sigma_\eta^2}{\sigma_d^2+\sigma_\eta^2}\delta$$

Example 3 (bounded memory) *In this example, we deal with a situation where the agents know the model but have a bounded memory in the sense they forget past realization of the shocks.*

Let's consider the problem of a firm which demand depends on expected aggregate demand and the price level. In order to keep things as simple as possible, we will assume that the price is an exogenous i.i.d process with mean $\overline{p}$ *and variance* σ_p^2*) and that aggregate demand is driven by the following simple AR(1) process*

$$Y_t = \rho Y_{t-1} + (1-\rho)\overline{Y} + \varepsilon_t$$

where $|\rho| < 1$ *and* ε_t *is the innovation of the process. The demand then takes the following form*

$$d_t = \alpha E(Y_{t+1}|\Omega) - \beta p_t$$

But rather than being defined as $\Omega = \{Y_{t-i}, p_{t-i}, \varepsilon_{t-i}; i = 0\ldots\infty\}$, Ω *now takes the form* $\Omega = \{Y_{t-i}, p_{t-i}, \varepsilon_{t-i}; i = 0\ldots k, k < \infty\}$. *Computing the rational expectation is now a bit more tricky. We first have to write down the Wold decomposition of the process of Y*

$$Y_t = \overline{Y} + \sum_{i=0}^{\infty}\rho^i\varepsilon_{t-i}$$

Then $E(Y_{t+1}|\Omega)$ *can be computed as*

$$E(Y_{t+1}|\Omega) = E\left(\overline{Y} + \sum_{i=0}^{\infty}\rho^i\varepsilon_{t+1-i}\middle|\Omega\right)$$

Since $\overline{Y}$ *is a deterministic constant,* $E\left(\overline{Y}\right) = \overline{Y}$, *such that*

$$E(Y_{t+1}|\Omega) = \overline{Y} + \sum_{i=0}^{\infty}\rho^i E(\varepsilon_{t+1-i}|\Omega)$$

Since $\Omega = \{Y_{t-i}, p_{t-i}, \varepsilon_{t-i}; i = 0\ldots k, k < \infty\}$, *we have* $\varepsilon_{t-i}\perp\Omega\ \forall i > k$, *such that, in this case* $E(\varepsilon_{t-i}|\Omega) = 0$. *Hence,*

$$E(Y_{t+1}|\Omega) = \overline{Y} + \sum_{i=0}^{k}\rho^{i+1}\varepsilon_{t-i}$$

hence

$$d_t = \alpha\left(\overline{Y} + \sum_{i=0}^{k} \rho^{i+1}\varepsilon_{t-i}\right) - \beta p_t$$

which may be re–expressed in terms of observable variables as

$$d_t = \alpha\left(\overline{Y} + \rho\left(Y_t - \overline{Y} - \rho^{k+1}\left(Y_{t-(k+1)} - \overline{Y}\right)\right)\right) - \beta p_t$$

1.2 A prototypical model of rational expectations

1.2.1 Sketching up the model

In this section we try to characterize the behavior of an endogenous variable y that obeys the following *expectational difference equation*

$$y_t = aE_t y_{t+1} + bx_t \tag{1.2}$$

where $E_t y_{t+1} \equiv E(y_{t+1}|\Omega)$ where $\Omega = \{y_{t-i}, x_{t-i}, i = 0, \ldots, \infty\}$.
Equation (1.2) may be given different interpretations. We now provide you with a number of models that suit this type of expectational difference equation.

Asset–pricing model: Let p_t be the price of a stock, d_t be the dividend, and r be the rate of return on a riskless asset, assumed to be held constant over time. Standard theory of finance teaches us that if agents are risk neutral, then the arbitrage between holding stocks and the riskless asset should be such that the expected return on the stock — given by the expected rate of capital gain plus the dividend/price ratio — should equal the riskless interest rate:

$$\frac{E_t p_{t+1} - p_t}{p_t} + \frac{d_t}{p_t} = r$$

or equivalently

$$p_t = aE_t p_{t+1} + ad_t \text{ where } a \equiv \frac{1}{1+r} < 1$$

The Cagan Model: The Cagan model is a macro model that was designed to furnish an explanation to the hyperinflation problem. Cagan assumes that the demand for real balances takes the following form

$$\frac{M_t^d}{P_t} = \exp\left(-\alpha\pi_{t+1}^e\right) \tag{1.3}$$

where π^e_{t+1} denotes expected inflation

$$\pi^e_{t+1} \equiv \frac{E_t(P_{t+1}) - P_t}{P_t}$$

In an equilibrium, money demand equals money supply, such that

$$M^d_t = M^s_t = M_t$$

hence in an equilibrium, equation (1.3) reduces to

$$\frac{M_t}{P_t} = \exp\left(-\alpha\frac{E_t(P_{t+1}) - P_t}{P_t}\right) \tag{1.4}$$

Taking logs — lowercases will denote logged variables — using the approximation $\log(1+x) \simeq x$ and reorganizing, we end up with

$$p_t = aE_t(p_{t+1}) + (1-a)m_t \text{ where } a = \frac{\alpha}{1+\alpha}$$

Monopolistic competition Consider a monopolist that faces the following demand

$$p_t = \alpha - \beta y_t - \gamma E_t y_{t+1} \tag{1.5}$$

the term in y_t accounts for the fact that the greater the greater the price is, the lower the demand is. The term in $E_t y_{t+1}$ accounts for the fact that greater expected sells tend to lower the price.[1] The firm acts as a monopolist maximizing its profit

$$\max_{y_t} p_t y_t - c_t y_t$$

taking the demand (1.5) into account. c_t is the marginal cost, which is assumed to follow an exogenous stochastic process. Note that we assume, for the moment, that the firm adopts a purely static behavior. Profit maximization — taking (1.5) into account — yields

$$\alpha - 2\beta y_t - \gamma E_t y_{t+1} - c_t = 0$$

which may be rewritten as

$$y_t = aE_t(p_{t+1}) + bc_t + d \text{ where } a = \frac{-\gamma}{2\beta},\ b = \frac{-1}{2\beta} \text{ and } d = \frac{\alpha}{2\beta}$$

[1] If $\gamma < 0$, the model may be given an alternative interpretation. Greater expected sells lead the firm to raise its price (you may think of goods such as tobacco, alcohol, ..., each good that may create addiction).

At this point we are left with the expectational difference equation (1.2), which may either be solved "forward" or "backward" looking depending on the value of a. When $|a| < 1$ the solution should be forward looking, as it will become clear in a moment, conversely, when $|a| > 1$ the model should be solved backward. The next section investigates this issue.

1.2.2 Forward looking solutions: $|a| < 1$

The problem that arises with the case $|a| < 1$ may be understood by looking at figure 1.1, which reports the dynamics of equation

$$E_t y_{t+1} = \frac{1}{a} y_t - \frac{b}{a} x_t$$

Holding x_t constant — and therefore eliminating the expectation. As can be seen from the figure, the path is fundamentally unstable as soon as we look at it in the usual backward looking way. Starting from an initial condition that differs from $\overline{y}$, say y_0, the dynamics of y diverges. The system then displays a bubble.[2] A more interesting situation arises when the variable y_t represents a variable such as a price or consumption — in any case a variable that shifts following a shock and that does not have an initial condition but a terminal condition of the form

$$\lim_{t \longrightarrow \infty} |y_t| < \infty \tag{1.6}$$

In fact such a terminal condition — which is often related to the so–called transversality condition arising in dynamic optimization models — bounds the sequence of $\{y_t\}_{t=0}^{\infty}$ and therefore imposes stationarity. Solving this system then amounts to find a sequence of stochastic variable that satisfies (1.2). This may be achieved in different ways and we now present 3 possible methods.

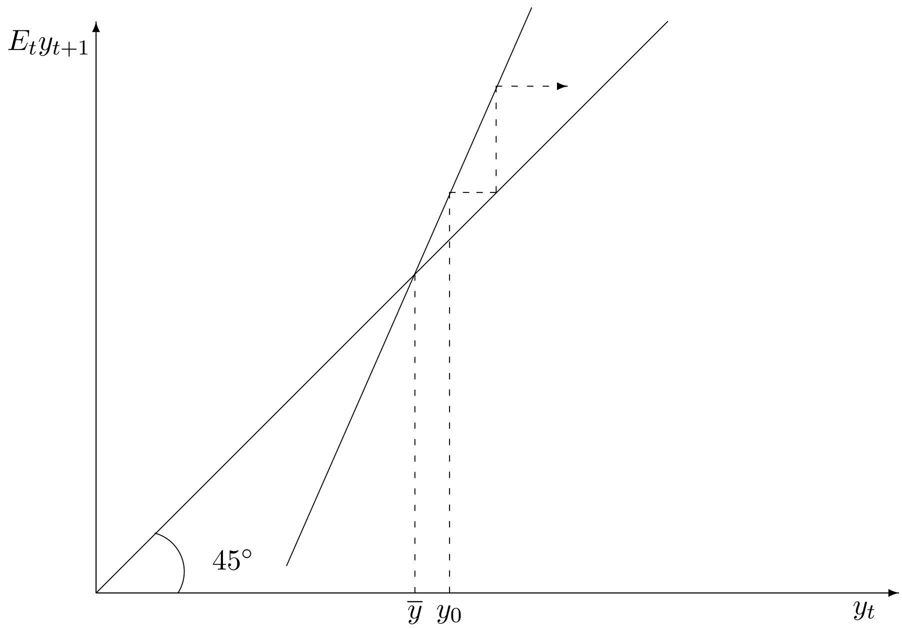
Forward substitution

This method proceeds by iterating forward on the system, making use of the law of iterated projection (proposition 3). Let us first recall the expectational difference equation at hand:

$$y_t = aE_t y_{t+1} + bx_t$$

[2] We will come back to this point later on.

Figure 1.1: The regular case



Iterating one step forward — that is plugging the value of y_t evaluated in $t+1$ in the expectation, we get

$$y_t = aE_t\left(E_{t+1}(ay_{t+2} + bx_{t+1})\right) + bx_t$$

The law of iterated projection implies that $E_t(E_{t+1}(y_{t+2})) = E_t y_{t+2}$, so that

$$y_t = a^2 E_t(y_{t+2}) + abE_t(x_{t+1}) + bx_t$$

Iterating one step forward, we get

$$y_t = a^2 E_t(E_{t+2}(ay_{t+3} + bx_{t+2})) + abE_t(x_{t+1}) + bx_t$$

Once again making use of the law of iterated projection, we get

$$y_t = a^3 E_t(y_{t+3}) + a^2 bE_t(x_{t+2}) + abE_t(x_{t+1}) + bx_t$$

Continuing the process, we get

$$y_t = b \lim_{k \longrightarrow \infty} \sum_{i=0}^{k} a^i E_t(x_{t+i}) + \lim_{k \longrightarrow \infty} a^{k+1} E_t(y_{t+k+1})$$

For the first term to converge, we need the expectation $E_t(x_{t+k})$ not to increase at a too fast pace. Then provided that $|a| < 1$, a sufficient condition for the first term to converge is that the expectation explodes at a rate lower than $|1/a-1|$.[3] In the sequel we will assume that this condition holds.
Finally, since $|a| < 1$, imposing that $\lim_{t\longrightarrow\infty} |y_t| < \infty$ holds, we have

$$\lim_{k\longrightarrow\infty} a^{k+1} E_t(y_{t+k+1}) = 0$$

and the solution is given by

$$y_t = b\sum_{i=0}^{\infty} a^i E_t(x_{t+i}) \qquad (1.7)$$

In other words, y_t is given by the discounted sum of all future expected values of x_t. In order to get further insight on the form of the solution, we may be willing to specify a particular process for x_t. We shall assume that it takes the following AR(1) form:

$$x_t = \rho x_{t-1} + (1-\rho)\overline{x} + \varepsilon_t$$

where $|\rho| < 1$ for sake of stationarity and ε_t is the innovation of the process. Note that

$$\begin{aligned}
E_t x_{t+1} &= \rho x_t + (1-\rho)\overline{x} \\
E_t x_{t+2} &= \rho E_t x_{t+1} + (1-\rho)\overline{x} = \rho^2 x_t + (1-\rho)(1+\rho)\overline{x} \\
E_t x_{t+3} &= \rho E_t x_{t+2} + (1-\rho)\overline{x} = \rho^3 x_t + (1-\rho)(1+\rho+\rho^2)\overline{x} \\
&\vdots \\
E_t x_{t+i} &= \rho^i x_t + (1-\rho)(1+\rho+\rho^2+\ldots+\rho^i)\overline{x} = \rho^i x_t + (1-\rho^{i+1})\overline{x}
\end{aligned}$$

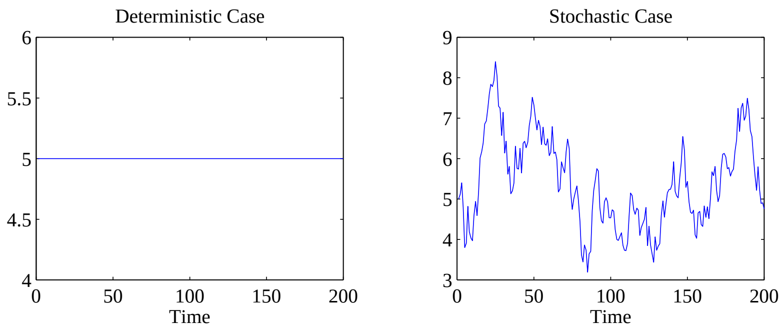
Therefore, the solution takes the form

$$\begin{aligned}
y_t &= b\sum_{i=0}^{\infty} a^i(\rho^i x_t + (1-\rho^i)\overline{x}) \\
&= b\left(\sum_{i=0}^{\infty}(a\rho)^i(x_t-\overline{x}) + \sum_{i=0}^{\infty} a^i\overline{x}\right) \\
&= b\left(\frac{x_t-\overline{x}}{1-a\rho} + \frac{\overline{x}}{1-a}\right) \\
&= \frac{b}{1-a\rho}x_t + \frac{ab(1-\rho)}{(1-a)(1-a\rho)}\overline{x}
\end{aligned}$$

[3] This will actually be the case with a stationary process.

Figure 1.2 provides an example of the process generated by such a solution, in the deterministic case and in the stochastic case. In the deterministic case, the economy always lies on its long–run value $y^\star$, which is the only stable point. We then talk about *steady state* — that is a situation where $y_t = y_{t+k} = y^\star$. In the stochastic case, the economy fluctuates around the mean of the process, and it is noteworthy that any change in x_t instantaneously translates into a change in y_t. Therefore, the persistence of y_t is given by that of x_t.

Figure 1.2: Forward Solution



Note: This example was generated using $a = 0.8$, $b = 1$, $\rho = 0.95$, $\sigma = 0.1$ and $\overline{x} = 1$.

Matlab Code: Forward Solution

```
\simple
%
% Forward solution
%
lg   = 100;
T    = [1:long];
a    = 0.8;
b    = 1;
rho  = 0.95;
sx   = 0.1;
xb   = 1;
%
% Deterministic case
%
y=a*b*xb/(1-a);
%
% Stochastic case
%
%
% 1) Simulate the exogenous process
%
x     = zeros(lg,1);
randn('state',1234567890);
```

```
e    = randn(lg,1)*sx;
x(1) = xb;
for i=2:long;
   x(i) = rho*x(i-1)+(1-rho)*xb+e(i);
end
%
% 2) Compute the solution
%
y    = b*x/(1-a*rho)+a*b*(1-rho)*xb/((1-a)*(1-a*rho));
```

Factorization

The method of factorization was introduced by Sargent [1979]. It amounts to make use of the forward operator F, introduced in the first chapter.[4] In a first step, equation (1.2) is rewritten in terms of F

$$y_t = aE_t y_{t+1} + bx_t \Longleftrightarrow E_t(y_t) = aE_t(y_{t+1}) + bE_t(x_t) \Longleftrightarrow (1-aF)E_t y_t = bE_t x_t$$

which rewrites as

$$E_t y_t = b\frac{E_t x_t}{1-aF}$$

since $|a|<1$, we have

$$\frac{1}{1-aF} = \sum_{i=0}^{\infty} a^i F^i$$

Therefore, we have

$$E_t y_t = y_t = b\sum_{i=0}^{\infty} a^i F^i E_t x_t = b\sum_{i=0}^{\infty} a^i E_t x_{t+i}$$

Note that although we get, obviously, the same solution, this method is not as transparent as the previous one since the terminal condition (1.6) does not appear explicitly.

Method of undetermined coefficients

This method proceeds by making an initial guess on the form of the solution. An educated guess for the problem at hand would be

$$y_t = \sum_{i=0}^{\infty} \alpha_i E_t x_{t+i}$$

[4] Recall that the forward operator is such that $F^i E_t(x_t) = E_t(x_{t+i})$.

Plugging the guess in (1.2) leads to

$$\sum_{i=0}^{\infty} \alpha_i E_t x_{t+i} = a E_t \left(\sum_{i=0}^{\infty} \alpha_i E_{t+1} x_{t+1+i} \right) + b x_t$$

Solving the model then amounts to find the sequence of α_i, $i = 0, \ldots, \infty$ such that the guess satisfies the equation. We then proceed by identification.

$$\begin{array}{ll} i=0 & \alpha_0 = b \\ i=1 & \alpha_1 = a\alpha_0 \\ i=2 & \alpha_2 = a\alpha_1 \\ \vdots & \end{array}$$

such that $\alpha_i = a\alpha_{i-1}$, with $\alpha_0 = b$. Note that since $|a| < 1$, this sequence converges toward 0 as i tends toward infinity. Therefore, the solution writes

$$y_t = b \sum_{i=0}^{\infty} a^i E_t x_{t+i}$$

The problem with such an approach is the we need to make the "right" guess from the very beginning. Assume for a while that we had specified the following guess

$$y_t = \gamma x_t$$

Then

$$\gamma x_t = a E_t \gamma x_{t+1} + b x_t$$

Identifying term by terms we would have obtained $\gamma = b$ or $\gamma = 0$, which is obviously a mistake.

As a simple example, let us assume that the process for x_t is given by the same AR(1) process as before. We therefore have to solve the following dynamic system

$$\left\{ \begin{array}{l} y_t = a E_t y_{t+1} + b x_t \\ x_t = \rho x_{t-1} + (1-\rho)\overline{x} + \varepsilon_t \end{array} \right.$$

Since the system is linear and that x_t exhibits a constant term, we guess a solution of the form

$$y_t = \alpha_0 + \alpha_1 x_t$$

Plugging this guess in the expectational difference equation, we get

$$\alpha_0 + \alpha_1 x_t = a E_t(\alpha_0 + \alpha_1 x_{t+1}) + b x_t$$

which rewrites, computing the expectation[5]

$$\alpha_0 + \alpha_1 x_t = a\alpha_0 + a\alpha_1 \rho x_t + a\alpha_1(1-\rho)\overline{x} + bx_t$$

Identifying term by term, we end up with the following system of equations

$$\alpha_0 = a\alpha_0 + a\alpha_1(1-\rho)\overline{x}\alpha_1 = a\alpha_1\rho + b$$

The second equation yields

$$\alpha_1 = \frac{b}{1-a\rho}$$

the first one gives

$$\alpha_0 = \frac{ab(1-\rho)}{(1-a)(1-a\rho)}\overline{x}$$

One advantage of this method is that it is particularly simple, and it requires the user to know enough on the economic problem to formulate the right guess. This latter property precisely constitutes the major drawback of the method as if formulating a guess is simple for linear economies it may be particularly tricky — even impossible — in all other cases.

1.2.3 Backward looking solutions: $|a| > 1$

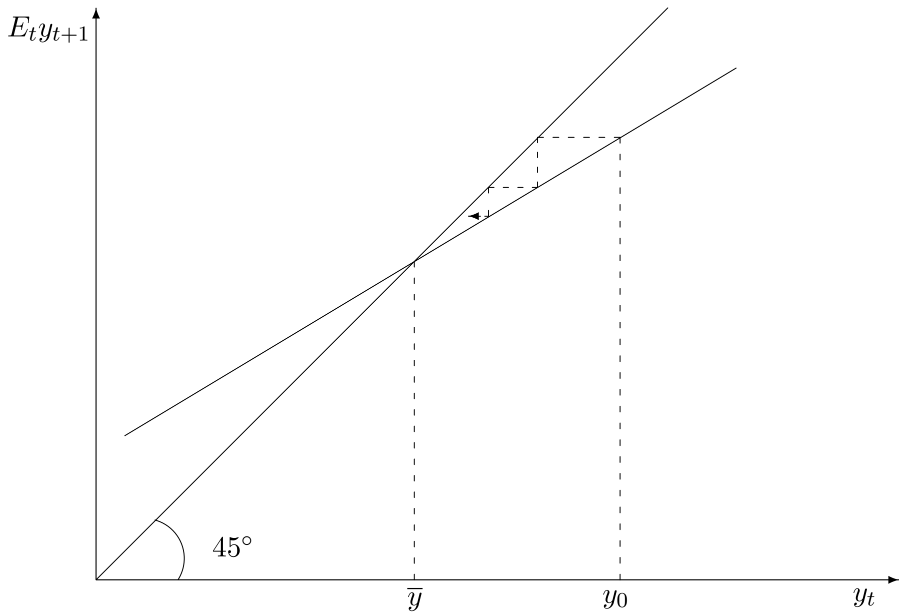
Until now, we have only considered the case of a regular economy in which $|a| < 1$, which — provided we are ready to impose a non–explosion condition — yields a unique solution that only involves fundamental shocks. In this section we investigate what happens when we relax the condition $|a| < 1$ and consider the case $|a| > 1$. This fundamentally changes the nature of the solution, as can be seen from figure 1.3. More precisely, any initial condition y_0 for y is admissible as any leads the economy back to its long–run solution $\overline{y}$. The equilibrium is then said to be indeterminate.

From a mathematical point of view, the sum involved in the forward solution is unlikely to converge. Therefore, the solution should be computed in an alternative way. Let us recall the expectational difference equation

$$y_t = aE_t y_{t+1} + bx_t$$

[5]Note that this is here that we make use of the assumptions on the process for the exogenous shock.

Figure 1.3: The irregular case



Note that, by construction, we have

$$y_{t+1} = E_t(y_{t+1}) + \zeta_{t+1}$$

where ζ_{t+1} is the expectational error, uncorrelated — by construction — with the information set, such that $E_t\zeta_{t+1} = 0$. The expectational difference equation then rewrites

$$y_t = a(y_{t+1} - \zeta_{t+1}) + bx_t$$

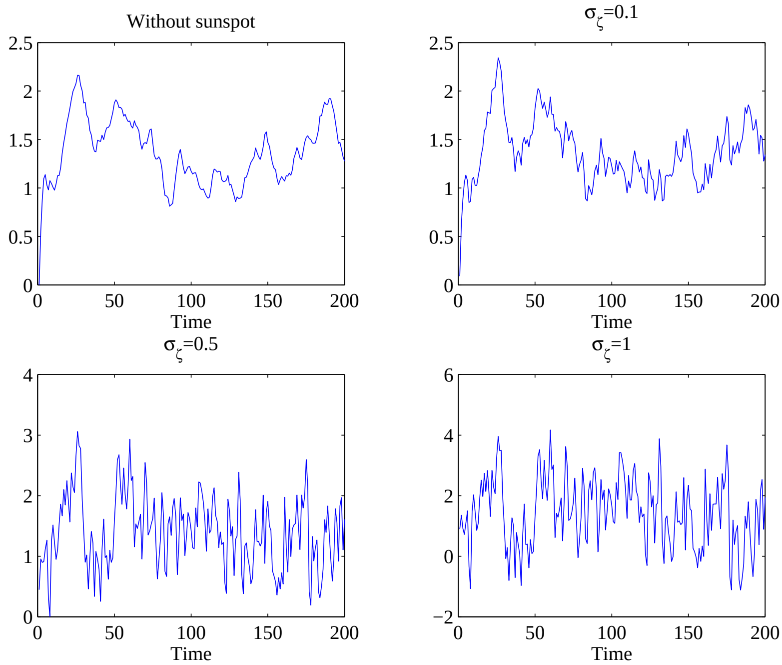
which may be restated as

$$y_{t+1} = \frac{1}{a}y_t + \frac{b}{a}x_t + \zeta_{t+1}$$

Since $|a| > 1$ this equation is stable and the system is fundamentally backward looking. Note that ζ_{t+1} is serially uncorrelated, and not necessarily correlated with the innovations of x_t. In other words, this shock may not be a fundamental shock and is alike a sunspot. For example, I wake up in the morning, look at the weather and decides to consume more. Why? I don't know! This is purely extrinsic to the economy!

Figure 1.4 reports an example of such an economy. We have drawn the solution to the model for different values of the volatility of the sunspot, using the same draw. As can be seen, although each solution is perfectly admissible, the properties of the economy are rather different depending on the volatility of the sunspot variable. Besides, one may compute the volatility and the first

Figure 1.4: Backward Solution



Note: This example was generated using $a = 1.8$, $b = 1$, $\rho = 0.95$, $\sigma = 0.1$ and $\overline{x} = 1$.

order autocorrelation of y_t:[6]

$$\begin{aligned}\sigma_y^2 &= \frac{b^2(\rho+a)}{(a^2-1)(a-\rho)}\sigma_x^2 + \frac{a^2}{a^2-1}\sigma_\zeta^2\\ \rho_y(1) &= \frac{1}{a}\left[1+\frac{b^2\rho(a^2-1)\sigma_x^2}{b^2(a+\rho)\sigma_x^2+a^2(a-\rho)\sigma_\zeta^2}\right]\end{aligned}$$

Therefore, as should be expected, the overall volatility of y is an increasing function of the volatility of the sunspot, but more important is the fact that its persistence is lower the greater the volatility of the sunspot. Hence, there

[6]We leave it to you as an exercize.

may be many candidates to the solution of such a backward looking equation, each displaying totally different properties.

Matlab Code: Backward Solution

```
%
% Backward solution
%
lg  = 200;
T   = [1:lg];
a   = 1.8;
b   = 1;
rho = 0.95;
sx  = 0.1;
xb  = 1;
se  = 0.1;
%
% 1) Simulate the exogenous process
%
x    = zeros(lg,1);
randn('state',1234567890);
e    = randn(lg,1)*sx;
x(1) = xb;
for i=2:lg;
   x(i) = rho*x(i-1)+(1-rho)*xb+e(i);
end
%
% 2) Compute the solution
%
randn('state',1234567891);
es = randn(lg,1);
y1  = zeros(lg,1); % without sunspot
y2  = zeros(lg,1); % with sunspot (se=0.1)
y3  = zeros(lg,1); % with sunspot (se=0.5)
y4  = zeros(lg,1); % with sunspot (se=1)
y1(1) = 0;
y2(1) = es(1)*0.1;
y3(1) = es(1)*0.5;
y4(1) = es(1);
for i=2:lg;
   y1(i) = y1(i-1)/a+b*x(i-1)/a;
   y2(i) = y2(i-1)/a+b*x(i-1)/a+0.1*es(i);
   y3(i) = y3(i-1)/a+b*x(i-1)/a+0.5*es(i);
   y4(i) = y4(i-1)/a+b*x(i-1)/a+es(i);
end
```

1.2.4 One step backward: bubbles

Let's now go back to the forward looking solution. The ways we dealt with it led us to eliminate any bubble — that is we imposed condition (1.6) to bound the sequence. By doing so, we restricted ourselves to a particular class of solution, but there may exist a wider class of admissible solution that satisfy (1.2) without being bounded.

Let us now assume that such an alternative solution of the form does exist

$$\widetilde{y}_t = y_t + b_t$$

where y_t is the solution (1.7) and b_t is a bubble. In order for $\widetilde{y}_t$ to be a solution to (1.2), we need to place some additional assumption on its behavior.

If $\widetilde{y}_t = y_t + b_t$ it has to be the case that $E_t\widetilde{y}_{t+1} = E_t y_{t+1} + E_t b_{t+1}$, such that plugging this in (1.2), we get

$$y_t + b_t = aE_t y_{t+1} + aE_t b_{t+1} + bx_t$$

Since y_t is a solution to (1.2), we have that $y_t = aE_t y_{t+1} + bx_t$ such that the latter equation reduces to

$$b_t = aE_t b_{t+1} \Longleftrightarrow E_t b_{t+1} = a^{-1} b_t$$

Therefore, any b_t that satisfies the latter restriction will be such that $\widetilde{y}_t$ is a solution to (1.2). Note that since $|a| < 1$ in the case of a forward solution, b_t explodes in expected values — therefore referring directly to the common sense of a speculative bubble. Up to this point we have not specified any particular functional form for the bubble. Blanchard and Fisher [1989] provide two examples of such bubbles:

1. The ever–expanding bubble: b_t then simply follows a deterministic trend of the form:

   $$b_t = b_0 a^{-t}$$

   It is then straightforward to verify that $b_t = aE_t b_{t+1}$. *How should we interpret such a behavior for the bubble?* In order to provide with some insights, let's consider the case of the asset–pricing equation:

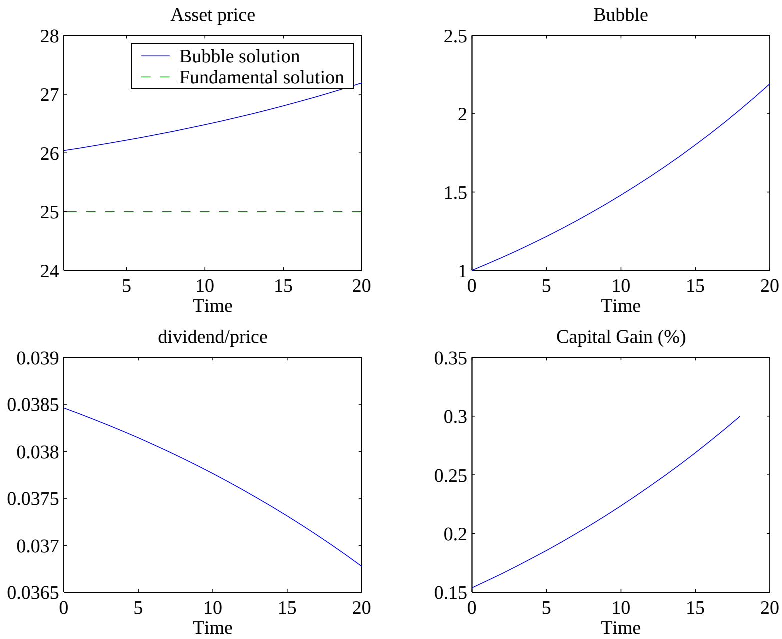
   $$\frac{E_t p_{t+1} - p_t}{p_t} + \frac{d_t}{p_t} = r$$

   where $d_t = d^\star$ (for simplicity). It is straightforward to check that the no–bubble solution (the fundamental solution) takes the form:

   $$p_t = p^\star = \frac{d^\star}{r}$$

which sticks to the standard solution that states that the price of an asset should be the discounted sum of expected dividends (you may check that $d^\star/r = \sum_{i=0}^{\infty}(1+r)^{-i}d^\star$). If we now add a bubble of the kind we consider — that is $b_t = b_0 a^{-t} = b_0(1+r)^t$ — provided $b_0 > 0$ the price of the asset will increase exponentially though the dividends are constant. The explanation for such a result is simple: individuals are ready to pay a price for the asset greater than expected dividends because they expect the price to be higher in future periods, which implies that expected capital gains will be able to compensate for the low price to dividend ratio. This kind of anticipation is said to be *self–fulfilling*. Figure 1.5 reports an example of such a bubble.

Figure 1.5: Deterministic Bubble



Note: This example was generated using $d^\star = 1$, $r = 0.04$.

MATLAB CODE: DETERMINISTIC BUBBLE

```
%
% Example of a deterministic bubble
% The case of asset pricing (constant dividends)
%
d_star = 1;
```

```
r      = 0.04;
%
% Fundamental solution p*
%
p_star = d_star/r;
%
% bubble
%
long   = 20;
T      = [0:long];
b      = (1+r).^T;
p      = p_star+b;
```

2. The bursting–bubble: A problem with the previous example is that the bubble is ever–expanding whereas observation and common sense suggests that sometimes the bubble bursts. We may therefore define the following bubble:

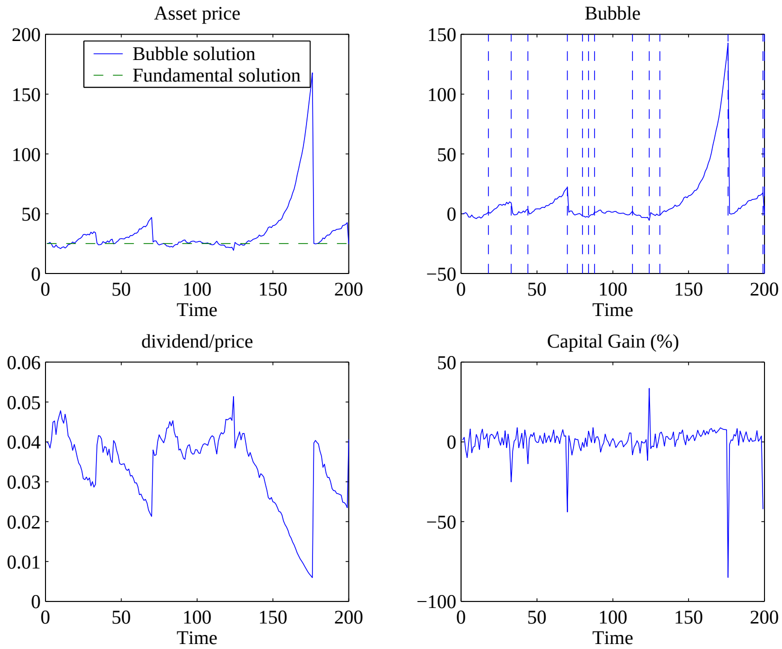
$$b_{t+1} = \begin{cases} (a\pi)^{-1}b_t + \zeta_{t+1} & \text{with probability } \pi \\ \zeta_{t+1} & \text{with probability } 1-\pi \end{cases}$$

with $E_t\zeta_{t+1} = 0$. So defined, the bubble keeps on inflating with probability π and bursts with probability $(1-\pi)$. Let's check that $b_t = aE_tb_{t+1}$

$$\begin{aligned} b_t &= aE_t(\pi((a\pi)^{-1}b_t + \zeta_{t+1}) + (1-\pi)\zeta_{t+1}) && \text{taking bursting into account} \\ &= aE_t(\pi(a\pi)^{-1}b_t) + \zeta_{t+1}) && \text{grouping terms in } \zeta_{t+1} \\ &= aE_t(a^{-1}b_t) && \text{since } E_t\zeta_{t+1} = 0 \\ &= b_t && \text{since } b_t \text{ is known in } t \end{aligned}$$

Figure 1.6 reports an example of such a bubble (the vertical lines in the upper right panel of the figure corresponds to time when the bubble bursts). The intuition for the result is the same as before: individuals are ready to pay a higher price for the asset than the expected discounted dividends because they expect with a sufficiently high probability that the price will be high enough in subsequent periods to generate sufficient capital gains to compensate for the lower price to dividend ratio. The main difference with the previous case is that this bubble is now driven by a stochastic variable, labelled as *sunspot* in the literature.

Figure 1.6: Bursting Bubble



Note: This example was generated using $d^{\star} = 1$, $\pi = 0.95$, $r = 0.04$.

Matlab Code: Bursting Bubble

```
%
% Example of a bursting bubble
% The case of asset pricing (constant dividends)
%
d_star = 1;
r      = 0.04;
%
% Fundamental solution p*
%
p_star = d_star/r;
%
% bubble
%
long   = 200;
prob   = 0.95;
randn('state',1234567890);
e      = randn(long,1);
rand('state',1234567890);
ind    = rand(long,1);
b      = zeros(long,1);
dum    = zeros(long,1);
b(1)   = 0;
for i  = 1:long-1;
   dum(i)= ind(i)<prob;
   b(i+1)= dum(i)*(b(i)*(1+r)/prob+e(i+1))+(1-dum(i))*e(i+1);
end;
p      = p_star+b;
```

Up to this point we have been dealing with very simple situations where the problem is either backward looking or forward looking. Unfortunately, such a case is rather scarce, and most of economic problems such as investment decisions, pricing decisions . . . are both backward and forward looking. We examine such situations in the next section.

1.3 A step toward multivariate Models

We are now interested in solving a slightly more complicated problem involving one lag (for the moment!) of the endogenous variable:

$$y_t = aE_t y_{t+1} + by_{t-1} + cx_t \tag{1.8}$$

This equation may be encountered in many different models, either in macro, micro, IO. . . as we will see later on. For the moment, let us assume that this is obtained from whatever model we may think of and let us take it as given.

We are now willing to solve this expectational equation. As before, there exist many methods.

1.3.1 The method of undetermined coefficients

Let us recall that solving the equation using undetermined coefficients amounts to formulate a guess for the solution and find some restrictions on the coefficients of the guess such that equation (1.8) is satisfied. An educated guess in this case is given by

$$y_t = \mu y_{t-1} + \sum_{i=0}^{\infty} \alpha_i E_t x_{t+i}$$

Where does this guess come from? **Experience!** and this is precisely why the method of undetermined coefficients, although it may appear particularly practical in a number of (simple) problems, is not always appealing.
Plugging this guess in equation (1.8) yields

$$\begin{aligned}
\mu y_{t-1} + \sum_{i=0}^{\infty} \alpha_i E_t x_{t+i} &= a E_t \left[\mu y_t + \sum_{i=0}^{\infty} \alpha_i E_{t+1} x_{t+1+i} \right] + b y_{t-1} + c x_t \\
&= a\mu \left(\mu y_{t-1} + \sum_{i=0}^{\infty} \alpha_i E_t x_{t+i} \right) + a E_t \left[\sum_{i=0}^{\infty} \alpha_i E_{t+1} x_{t+1+i} \right] \\
&\quad + b y_{t-1} + c x_t \\
&= (a\mu^2 + b) y_{t-1} + a\mu \sum_{i=0}^{\infty} \alpha_i E_t x_{t+i} + a \sum_{i=0}^{\infty} \alpha_i E_t x_{t+1+i} + c x_t
\end{aligned}$$

Everything is then a matter of identification (term by term):

$$\mu = a\mu^2 + b \tag{1.9}$$

$$\alpha_0 = a\mu\alpha_0 + c \tag{1.10}$$

$$\alpha_i = a\mu\alpha_i + a\alpha_{i-1} \quad \forall i \geqslant 1 \tag{1.11}$$

Solving (1.9) for μ amounts to solve the second order polynomial

$$\mu^2 - \frac{1}{a}\mu + \frac{b}{a} = 0$$

which admits two solutions such that

$$\begin{cases} \mu_1 + \mu_2 = \frac{1}{a} \\ \mu_1 \mu_2 = \frac{b}{a} \end{cases}$$

Three configurations may emerge from the above equation

1. the two solutions lie outside the unit circle: the model is said to be a source and only one particular point — the steady state — is a solution to the equation.

2. One solution lie outside the unit circle and the other one inside: the model exhibits the saddle path property.

3. The two solutions lie inside the unit circle: the model is said to be a sink and there is indeterminacy.

Here, we will restrict ourselves to the situation where an extended version of the condition $|a| < 1$ we were dealing with in the preceding section holds, namely one root will be of modulus greater than one and the other less than one. The model will therefore exhibit the so–called *saddle point property*, for which we will provide a geometrical interpretation in a moment. To sum up, we consider a situation where $|\mu_1| < 1$ and $|\mu_2| > 1$. Since we restrict ourselves to the stationary solution, we necessarily have $|\mu| < 1$ so that $\mu = \mu_1$.

Once μ has been obtained, we can solve for α_i, $i = 0, \ldots$. α_0 is obtained from (1.10) and takes the value

$$\alpha_0 = \frac{c}{1 - a\mu_1}$$

We then get α_i, $i \geqslant 1$, from (1.11) as

$$\alpha_i = \frac{a}{1 - a\mu_1}\alpha_{i-1} = \frac{1}{\frac{1}{a} - \mu_1}\alpha_{i-1}$$

Since $\mu_1 + \mu_2 = 1/a$, the latter equation rewrites

$$\alpha_i = \mu_2^{-1}\alpha_{i-1}$$

where $|\mu_2| > 1$, such that this sequence converges toward zero. Therefore the solution is given by

$$y_t = \mu_1 y_{t-1} + \frac{c}{1 - a\mu_1}\sum_{i=0}^{\infty}\mu_2^{-i}E_t x_{t+i}$$

Example 4 *In the case of an AR(1) process for x_t, the solution is straightforward, as all the process may be simplified. Indeed, let us consider the following problem*

$$\left\{\begin{array}{l} y_t = aE_t y_{t+1} + by_{t-1} + cx_t \\ x_t = \rho x_{t-1} + \varepsilon_t \end{array}\right.$$

with $\varepsilon_t \rightsquigarrow \mathcal{N}(0,\sigma)$. *An educated guess for the solution of this equation would be*

$$y_t = \mu y_{t-1} + \alpha x_t$$

Let us then compute the solution of the problem, that is let us find μ *and* α. *Plugging the guess for the solution in the expectational difference equation leads to*

$$\begin{aligned}
\mu y_{t-1} + \alpha x_t &= aE_t(\mu y_t + \alpha x_{t+1}) + by_{t-1} + cx_t \\
&= a\mu^2 y_{t-1} + a\mu\alpha x_t + a\alpha\rho x_t + by_{t-1} + cx_t \\
&= (a\mu^2 + b)y_{t-1} + (c + a\alpha(\mu+\rho))x_t
\end{aligned}$$

Therefore, we have to solve the system

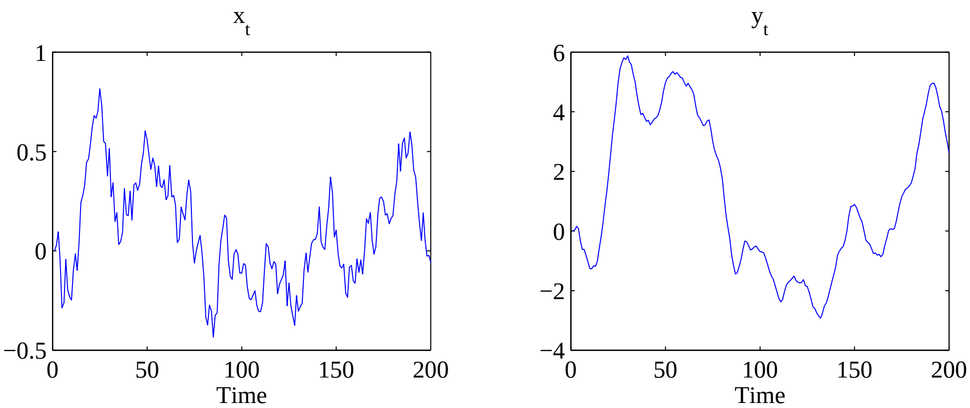
$$\begin{cases} \mu = a\mu^2 + b \\ \alpha = c + a\alpha(\mu+\rho) \end{cases}$$

Like in the general case, we select the stable root of the first equation μ_1, *such that* $|\mu_1| < 1$, *and* $\alpha = \frac{c}{1-a(\mu_1+\rho)}$ *Figure (1.7) reports an example of such an economy for two different parameterizations.*

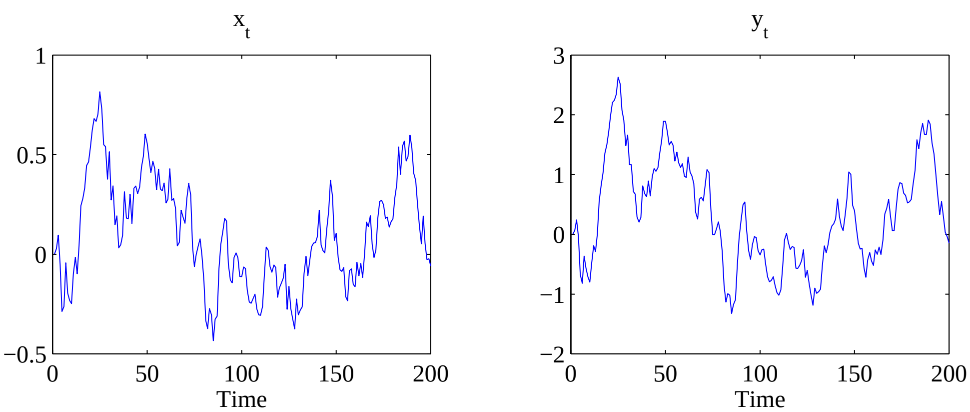
Matlab Code: Backward–Forward Solution

```
%
% Solve for
%
% y(t)=a E y(t+1) + b y(t-1) + c x(t)
% x(t)= rho x(t-1)+e(t)    e iid(0,se)
%
% and simulate the economy!
%
a      = 0.25;
b      = 0.7;
c      = 1;
rho    = 0.95;
se     = 0.1;

mu     = roots([a -1 b]);
[m,i]  = min(mu);
mu1    = mu(i);
[m,i]  = max(mu);
mu2    = mu(i);

alpha = b/(1-a*(mu1+rho));
%
% Simulation
%
```

Figure 1.7: Backward–forward solution



Note: $a = 0.25$, $b = 0.7$, $\rho = 0.95$, $\sigma = 0.1$



Note: $a = 0.7$, $b = 0.25$, $\rho = 0.95$, $\sigma = 0.1$

```
lg     = 200;
randn('state',1234567890);
e      = randn(lg,1)*se;
x      = zeros(lg,1);
y      = zeros(lg,1);
x(1)   = 0;
y(1)   = alpha*x(1);
for i = 2:lg;
   x(i) = rho*x(i-1)+e(i);
   y(i) = mu1*y(i-1)+alpha*x(i);
end
```

Note that contrary to the simple case we considered in the previous section, the solution does not only inherit the persistence of the shock, but also generates its own persistence through μ_1 as can be seen from the first order autocorrelation

$$\rho(1) = \frac{\mu_1 + \rho}{1 + \mu_1 \rho}$$

1.3.2 Factorization

The method of factorization proceeds into 2 steps.

1. Factor the model (1.8) making use of the leading operator F:

$$(aF^2 - F + b)E_t y_{t-1} = -cE_t x_t$$

   which may be rewritten as

$$\left(F^2 - \frac{1}{a}F + \frac{b}{a}\right) E_t y_{t-1} = -\frac{c}{a} E_t x_t$$

   which may also be rewritten as

$$(F - \mu_1)(F - \mu_2)E_t y_{t-1} = -\frac{c}{a} E_t x_t$$

   Note that μ_1 and μ_2 are the same as the ones obtained using the method of undetermined coefficients, therefore the same discussion about their size applies. We restrict ourselves to the case $|\mu_1| < 1$ (backward part) and $|\mu_2| > 1$ (forward part) — that is to saddle path solutions.

2. Derive a solution for y_t: Starting from the last equation, we can rewrite it as

$$(F - \mu_1)E_t y_{t-1} = -\frac{c}{a}(F - \mu_2)^{-1} E_t x_t$$

or

$$(F - \mu_1)E_t y_{t-1} = \frac{c}{a\mu_2}(1 - \mu_2^{-1}F)^{-1}E_t x_t$$

Since $|\mu_2| > 1$, we know that

$$(1 - \mu_2^{-1}F)^{-1} = \sum_{i=0}^{\infty} \mu_2^{-i} F^i$$

so that

$$(F - \mu_1)E_t y_{t-1} = \frac{c}{a\mu_2}\sum_{i=0}^{\infty} \mu_2^{-i} F^i E_t x_t = \frac{c}{a\mu_2}\sum_{i=0}^{\infty} \mu_2^{-i} E_t x_{t+i}$$

Now, applying the leading operator on the left hand side of the equation and acknowledging that $\mu_2 = 1/a - \mu_1$, we have

$$y_t = \mu_1 y_{t-1} + \frac{c}{1 - a\mu_1}\sum_{i=0}^{\infty} \mu_2^{-i} E_t x_{t+i}$$

1.3.3 A matricial approach

In this section, we would like to provide you with some geometrical intuition of what is actually going on when the saddle path property applies in the model. To do so, we will rely on a matricial approach. First of all, let us recall the problem we have in hands:

$$y_t = aE_t y_{t+1} + by_{t-1} + cx_t$$

Introducing the technical variable z_t defined as

$$z_{t+1} = y_t$$

the model may be rewritten as[7]

$$\begin{pmatrix} E_t y_{t+1} \\ z_{t+1} \end{pmatrix} = \begin{pmatrix} \frac{1}{a} & -\frac{b}{a} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_t \\ z_t \end{pmatrix} - \begin{pmatrix} -c \\ 1 \end{pmatrix} x_t$$

Remember that $E_t y_{t+1} = y_{t+1} - \zeta_{t+1}$ where ζ_{t+1} is an iid process which represents the expectation error, therefore, the system rewrites

$$\begin{pmatrix} y_{t+1} \\ z_{t+1} \end{pmatrix} = \begin{pmatrix} \frac{1}{a} & -\frac{b}{a} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_t \\ z_t \end{pmatrix} - \begin{pmatrix} -c \\ 1 \end{pmatrix} x_t - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \zeta_{t+1}$$

[7] In the next section we will actually pool all the equations in a single system, but for pedagogical purposes let us separate exogenous variables from the rest for a while.

In order to understand the saddle path property let us focus on the homogenous part of the equation

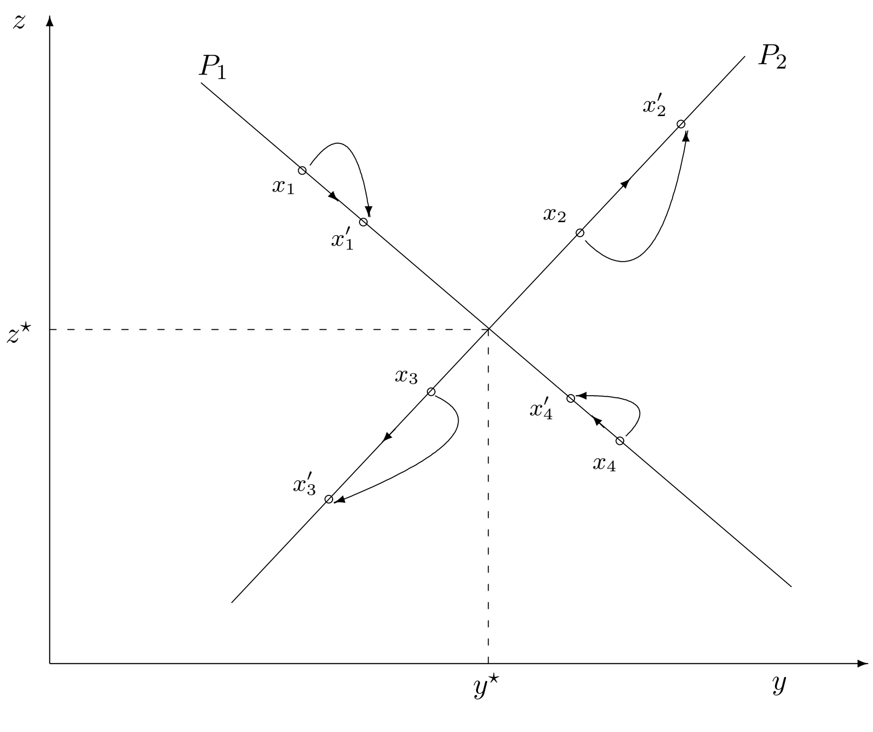
$$\begin{pmatrix} y_{t+1} \\ z_{t+1} \end{pmatrix} = \begin{pmatrix} \frac{1}{a} & -\frac{b}{a} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_t \\ z_t \end{pmatrix} = W \begin{pmatrix} y_t \\ z_t \end{pmatrix}$$

Provided $b \neq 0$ the matrix W can be diagonalized and may be rewritten as

$$W = PDP^{-1}$$

where D contains the two eigenvalues of W and P the associated eigenvectors. Figure 1.8 provides a way of thinking about eigenvectors and eigenvalues in dynamical systems. The figure reports the two eigenvectors, P_1 and P_2, associated with the two eigenvalues μ_1 and μ_2 of W. μ_1 is the stable root and μ_2 is the unstable root. As can be seen from the graph, displacements along

Figure 1.8: Geometrical interpretation of eigenvalues/eigenvectors



P_1 are convergent, in the sense they shift either x_1 or x_4 toward the center of the graph (x'_1 and x'_4), while displacements along P_2 are divergent (shift of x_2 and x_3 to x'_2 and x'_3). In fact the eigenvector determines the direction along which the system will evolve and the eigenvalue the speed at which the shift will take place.

The characteristic equation that gives the eigenvalues, in the case we are studying, is given by

$$det(W - \mu I) = 0 \Longleftrightarrow \mu^2 - \frac{1}{a}\mu + \frac{b}{a} = 0$$

which exactly corresponds to the equations we were dealing with in the previous sections. We will not enter the formal resolution of the model right now, as we will undertake an extensive treatment in the next section. However, we will just try to understand what may be going on using a phase diagram like approach to understand the dynamics. Figures 1.9–1.11 report the different possible configuration we may encounter solving this type of model. The first one is a *source* (figure 1.9), which is such that no matter the initial condition we feed the system with — except $y_0 = y^\star$, $z_0 = z^\star$ — the system will explode. Both y and z will not be bounded. The second one is a *sink* (figure 1.10), all trajectories converge back to the steady state of the economy, one is then free to choose whatever trajectory it wants to go back to the steady state. The equilibrium is therefore indeterminate.

Figure 1.9: A source

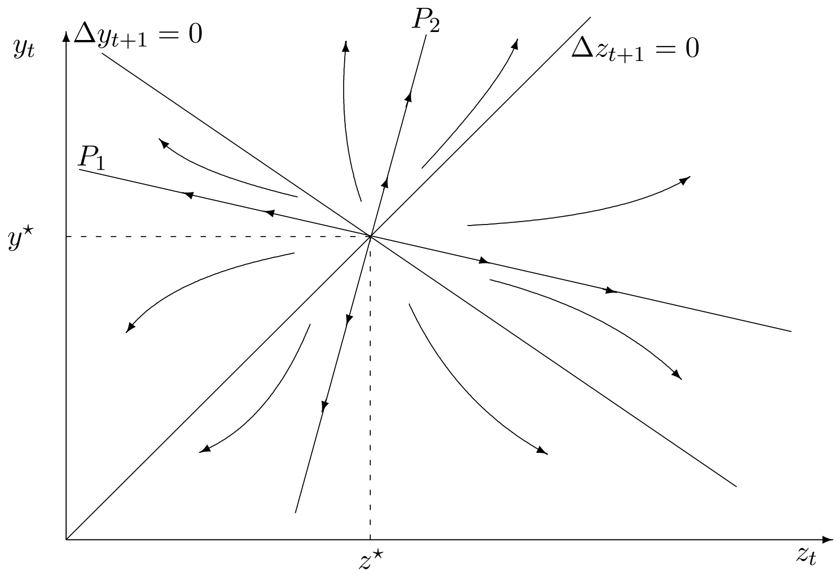
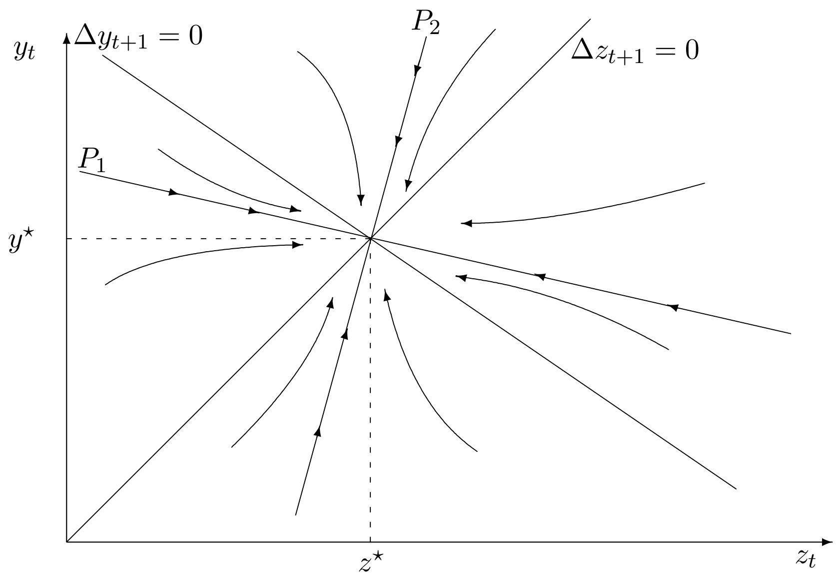


Figure 1.10: A sink: indeterminacy

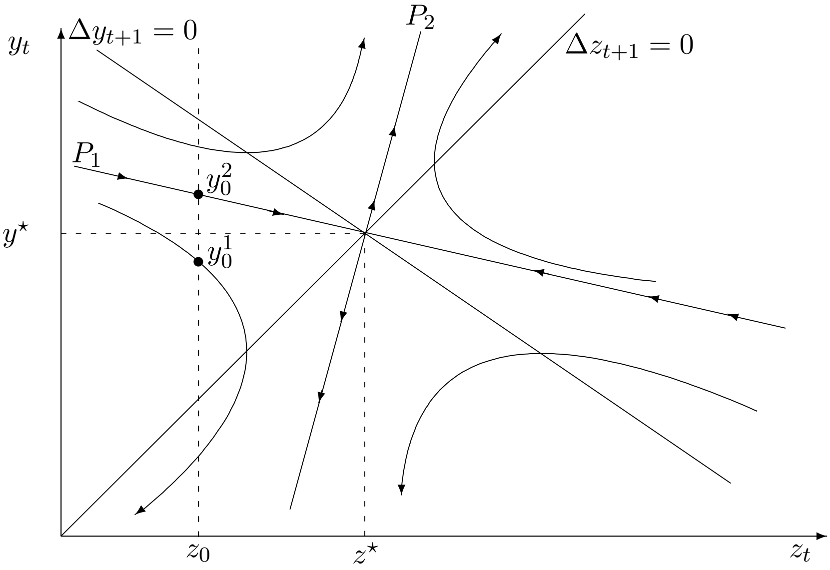


In the last situation (figure 1.11) — this corresponds to the most commonly encountered situation in economic theory — the economy lies on a saddle: one branch of the saddle converges to the steady state, the other one diverges. The problem is then to select where to start from. It should be clear to you that in t, z_t is perfectly known as $z_t = y_{t-1}$ which was selected in the earlier period. z_t is then said to be predetermined: the agents is endowed with its value when she enters the period. This is part of the information set. Solving the system therefore amounts to select a value for y_t, given that for z_t and the structure of the model. *How to proceed then?* Let us assume for a while that at time 0, the economy is endowed with z_0, and assume that we impose the value y_0^1 as a starting value for y. In such a case, the economy will explode: in other words a solution including a bubble has been selected. If, alternatively, y_0^2 is selected, then the economy will converge to the steady state $(z^\star, y^\star)$ and all the variables will be bounded. In other words, we have selected a trajectory such that

$$\lim_{t \longrightarrow \infty} |y_t| < \infty$$

holds. Otherwise stated, bubbles have been eliminated by imposing a terminal condition. In the sequel, we will be mostly interested by situation were the

Figure 1.11: The saddle path



economy either lies on a saddle path or is indeterminate. In the next section, we will show you how to solve an expectational multivariate system of the kind we were considering up to now.

1.4 Multivariate Rational Expectations Models (The simple case)

1.4.1 Representation

Let us assume that the model writes

$$M_{cc}\mathcal{Y}_t = M_{cs}M_{cs}\mathcal{S}_t \quad (1.12)$$

$$M_{ss0}E_t\mathcal{S}_{t+1} + M_{ss1}\mathcal{S}_t = M_{sc0}E_t\mathcal{Y}_{t+1} + M_{sc1}\mathcal{Y}_t + M_{se}\mathcal{E}_{t+1} \quad (1.13)$$

where Y_t is a $n_y \times 1$ vector of endogenous variables, ε_t is a $\ell \times 1$ vector of exogenous serially uncorrelated random disturbances. A fairly natural interpretation of this dynamic system may be found in the state–space form literature: equation (1.17) corresponds to the standard measurement equation. It relates variables of interest $\mathcal{Y}_t$ to state variables $\mathcal{S}_t$. (1.13) is the state equation that actually drives the dynamics of the economy under consideration:[8] it relates future values of states $\mathcal{S}_{t+1}$ to current and expected values of variables of interest, current state variables and shocks to fundamentals $\mathcal{E}_{t+1}$. In other words, (1.13) furnishes the transition from one state of the system to another one. Our problem is then to solve this system.

As a first step, it would be great if we were able to eliminate all variables defined by the measurement equation and restrict ourselves to a state equation, as it would bring us back to our initial problem. To do so, we use (1.17) to eliminate $\mathcal{Y}_t$.

$$\mathcal{Y}_t = M_{cc}^{-1} M_{cs} \mathcal{S}_t$$

Plugging this expression in (1.13), we obtain:

$$E_t \mathcal{S}_{t+1} = W_S\, \mathcal{S}_t + W_E \mathcal{E}_{t+1}$$

where

$$\begin{aligned} W_S &= -\left(M_{ss0} - M_{sc0}\, M_{cc}^{-1}\, M_{cs}\right)^{-1} \left(M_{ss1} - M_{sc1}\, M_{cc}^{-1}\, M_{cs}\right) \\ W_E &= \left(M_{ss0} - M_{sc0}\, M_{cc}^{-1}\, M_{cs}\right)^{-1} Mse \end{aligned}$$

We are then back to our expectational difference equation. But it needs additional work. Indeed, Farmer proposes a method that enables us to forget about expectations when solving for the system. He proposes to replace the expectation by the actual variable minus the expectation error

$$E_t \mathcal{S}_{t+1} = \mathcal{S}_{t+1} - \mathcal{Z}_{t+1}$$

where $E_t \mathcal{Z}_{t+1} = 0$. Then the system rewrites

$$\mathcal{S}_{t+1} = W_S\, \mathcal{S}_t + W_E \mathcal{E}_{t+1} + \mathcal{Z}_{t+1} \tag{1.14}$$

This is the system we will be dealing with.

[8] Let us accept that statement for the moment, things will become clear as we will move to examples.

1.4.2 Solving the system

? have shown that the existence and uniqueness of a solution depends fundamentally on the position of the eigenvalues of W_S relative to the unit circle. Denoting by N_B and N_F the number of, respectively, predetermined and jump variables, and by N_I and N_O the number of eigenvalues that lie inside and outside the unit circle, we have the following proposition.

Proposition 4

(i) If $N_I = N_B$ and $N_O = N_F$, then there exists a unique solution path for the rational expectation model that converges to the steady state;

(ii) If $N_I > N_B$ (and $N_O < N_F$), then the system displays indeterminacy;

(iii) If $N_I > N_B$ (and $N_O > N_F$), then the system is a source.

Hereafter we will deal with the two first situations, the last one being never studied in economics.

The diagonalization of W_S leads to

$$W_S = P\,D\,P^{-1}$$

where D is the matrix that contains the eigenvalues of W_S on its diagonal and P is the matrix that contains the associated eigenvectors. For convenience, we assume that both D and P are such that eigenvalues are sorted in the ascending order. We shall then consider two cases

1. The model satisfies the saddle path property ($N_I = N_B$ and $N_O = N_F$)
2. The model exhibit indeterminacy ($N_I > N_B$ and $N_O < N_F$)

The saddle path

In this section, we consider the case were the model satisfies the saddle path property ($N_I = N_B$ and $N_O = N_F$). For convenience, we consider the following partitioning of the matrices

$$D = \begin{pmatrix} D_B & 0 \\ 0 & D_F \end{pmatrix},\; P = \begin{pmatrix} P_{BB} & P_{BF} \\ P_{FB} & P_{FF} \end{pmatrix},\; P^{-1} = \begin{pmatrix} P^\star_{BB} & P^\star_{BF} \\ P^\star_{FB} & P^\star_{FF} \end{pmatrix}$$

This partition conforms the position of the eigenvalues relative to the unit circle. For instance, a B stands for the set of eigenvalues that lie within the unit circle, whereas B stands for the set of eigenvalues that lie out of it.
We then apply the following modification to the system in order to make it diagonal:

$$\widetilde{\mathcal{S}}_t = P^{-1}\mathcal{S}_t$$

so that

$$P^{-1}\mathcal{S}_{t+1} = P^{-1}W_S P\, P^{-1}\mathcal{S}_t + P^{-1}W_E\mathcal{E}_{t+1} + P^{-1}\mathcal{Z}_{t+1}$$

or

$$\widetilde{\mathcal{S}}_{t+1} = D\,\widetilde{\mathcal{S}}_t + R\,\mathcal{E}_{t+1} + P^{-1}\mathcal{Z}_{t+1}$$

The same partitioning is applied to R

$$R = \begin{pmatrix} R_{B.} \\ R_{F.} \end{pmatrix}$$

and the state vector

$$\widetilde{\mathcal{S}}_t = \begin{pmatrix} \widetilde{\mathcal{S}}_{B,t} \\ \widetilde{\mathcal{S}}_{F,t} \end{pmatrix}$$

The system then rewrites as

$$\begin{pmatrix} \widetilde{\mathcal{S}}_{B,t+1} \\ \widetilde{\mathcal{S}}_{F,t+1} \end{pmatrix} = \begin{pmatrix} D_B & 0 \\ 0 & D_F \end{pmatrix}\begin{pmatrix} \widetilde{\mathcal{S}}_{B,t} \\ \widetilde{\mathcal{S}}_{F,t} \end{pmatrix} + \begin{pmatrix} R_{B.} \\ R_{F.} \end{pmatrix}\mathcal{E}_{t+1} + \begin{pmatrix} P^{\star}_{B.} \\ P^{\star}_{F.} \end{pmatrix}\mathcal{Z}_{t+1}$$

Therefore, the law of motion of forward variables is given by

$$\widetilde{\mathcal{S}}_{F,t+1} = D_F\widetilde{\mathcal{S}}_{F,t} + R_{F.}\mathcal{E}_{t+1} + P^{\star}_{F.}\mathcal{Z}_{t+1}$$

Taking expectations on both side of the equation

$$E_t\widetilde{\mathcal{S}}_{F,t+1} = D_F\widetilde{\mathcal{S}}_{F,t} \Longleftrightarrow \widetilde{\mathcal{S}}_{F,t} = D_F^{-1}E_t\widetilde{\mathcal{S}}_{F,t+1}$$

since D_F is a diagonal matrix, forward iteration yields

$$\widetilde{\mathcal{S}}_{F,t} = \lim_{j\to\infty} D_F^{-j}E_t\widetilde{\mathcal{S}}_{F,t+j}$$

Provided $E_t\widetilde{\mathcal{S}}_{F,t+j}$ is bounded — which amounts to eliminate bubbles — we have

$$\lim_{j\to\infty} D_F^{-j}E_t\widetilde{\mathcal{S}}_{F,t+j} = 0 \Longleftrightarrow \widetilde{\mathcal{S}}_{F,t} = 0$$

Then by construction, we have

$$\widetilde{\mathcal{S}}_{F,t} = P^{\star}_{FB}\mathcal{S}_{B,t} + P^{\star}_{FF}\mathcal{S}_{F,t}$$

which furnishes a restriction on $\mathcal{S}_{B,t}$ and $\mathcal{S}_{F,t}$

$$P^{\star}_{FB}\mathcal{S}_{B,t} + P^{\star}_{FF}\mathcal{S}_{F,t} = 0$$

This condition expresses the relationship that relates the jump variables to the predetermined variables, and therefore defined the initial condition $S_{F,t}$ which is compatible with *(i)* the initial conditions on the predetermined variables and *(ii)* the stationarity of the solution:

$$\mathcal{S}_{F,t} = -(P^{\star}_{FF})^{-1}P^{\star}_{FB}\mathcal{S}_{B,t} = \Gamma\mathcal{S}_{B,t}$$

Plugging this result in the law of motion of backward variables we have

$$\mathcal{S}_{B,t+1} = (W_{BB} + W_{BF}\Gamma)\mathcal{S}_{B,t} + R_B\mathcal{E}_{t+1} + \mathcal{Z}_{B,t+1}$$

but by definition, no expectation error may be done when predicting a predetermined variable, such that $\mathcal{Z}_{Bt+1} = 0$. Hence, the solution of the problem is given by

$$\mathcal{S}_{B,t+1} = M_{SS}\mathcal{S}_{B,t} + M_{SE}\mathcal{E}_{t+1} \tag{1.15}$$

where $M_{SS} = (W_{BB} + W_{BF}\Gamma)$ and $M_{SE} = R_B$.
As far as the measurement equation is concerned, thing are then rather simple. Let us define $\Phi = M_{cc}^{-1}M_{cs} = (\Phi_B \vdots \Phi_F)$, we have

$$\mathcal{Y}_t = \Phi_B\mathcal{S}_{B,t} + \Phi_F\mathcal{S}_{F,t} = \Pi\mathcal{S}_{B,t}$$

where $\Pi = (\Phi_B + \Phi_F\Gamma)$.
The system is therefore solved and may be represented as

$$\mathcal{S}_{B,t+1} = M_{SS}\mathcal{S}_{B,t} + M_{SE}\mathcal{E}_{t+1} \tag{1.16}$$
$$\mathcal{Y}_t = \Pi\mathcal{S}_{B,t} \tag{1.17}$$
$$\mathcal{S}_{F,t} = \Gamma\mathcal{S}_{B,t} \tag{1.18}$$

1.5 Multivariate Rational Expectations Models (II)

In this section we present **a** method to solve for multivariate rational expectations models, "**a**" because there are many of them (almost as many as authors that deal with this problem).[9] The one we present was introduced by Sims [2000] and recently revisited by Lubik and Schorfheide [2003]. It has the advantage of being general and explicitly dealing with expectation errors. This latter property makes it particularly suitable for solving sunspot equilibria.

1.5.1 Preliminary Linear Algebra

Generalized Schur Decomposition: This is a method to obtain eigenvalues from a system which is not invertible. One way to think of this approach is to remember that when we compute the eigenvalues of a diagonalizable matrix A, we want to find a numberλ and an associated eigenvector V such that

$$(A-\lambda I)V=0$$

The generalized Schur decomposition of two matrices A and B attempts to compute something similar, but rather than considering $(A-\lambda I)$, the problem considers $(A-\lambda B)$. A more formal, and — above all — a more rigorous statement of the Schur decomposition is given by the following definitions and theorem.

Definition 4 *Let $P \in \mathbb{C} \longrightarrow \mathbb{C}^{n\times n}$ be a matrix–valued function of a complex variable (a matrix pencil). Then the set of its generalized eigenvalues $\lambda(P)$ is defined as*

$$\lambda(P)=\{z\in\mathbb{C} : |P(z)=0\}$$

When $P(z)$ writes as $Az-B$, we denote this set as $\lambda(A,B)$. Then there exists a vector V such that $BV=\lambda AV$.

Definition 5 *Let $P(z)$ be a matrix pencil, P is said to be regular if there exists $z\in\mathbb{C}$ such that $|P(z)|\neq 0$ — i.e. if $\lambda(P)\neq\mathbb{C}$.*

[9] In the appendix we present an alternative method that enables you to solve for singular systems.

Theorem 1 (The complex generalized Schur form) *Let A and B belong to $\mathbb{C}^{n\times n}$ and be such that $P(z) = Az - B$ is a regular matrix pencil. Then there exist unitary $n \times n$ matrices of complex numbers Q and Z such that*

1. *$S = Q'AZ$ is upper triangular,*
2. *$T = Q'BZ$ is upper triangular,*
3. *For each i, S_{ii} and T_{ii} are not both zero,*
4. *$\lambda(A, B) = \{T_{ii}/S_{ii} : S_{ii} \neq 0\}$*
5. *The pairs (T_{ii}, S_{ii}), $i = 1 \ldots n$ can be arranged in any order.*

A formal proof of this theorem may be found in Golub and Van Loan [1996].

Singular Value Decomposition: The singular value decomposition is used for non–square matrices and is the most general form of diagonalization. Any complex matrix $A(n \times m)$ can be factored into the form

$$A = UDV'$$

where $U(n \times n)$, $D(n \times m)$ and $V(m \times m)$, with U and V unitary matrices ($UU' = VV' = I_{(n\times n)}$). D is a diagonal matrix with positive values d_{ii}, $i = 1 \ldots r$ and 0 elsewhere. r is the rank of the matrix. d_{ii} are called the singular values of A.

1.5.2 Representation

Let us assume that the model writes

$$A_0 Y_t = A_1 Y_{t-1} + B\varepsilon_t + C\eta_t \tag{1.19}$$

where Y_t is a $n \times 1$ vector of endogenous variables, ε_t is a $\ell \times 1$ vector of exogenous serially uncorrelated random disturbances, and η_t is a $k \times 1$ vector of expectation errors satisfying $E_{t-1}\eta_t = 0$ for all t. A_0 and A_1 are both $n \times n$ coefficient matrices, while B is $n \times \ell$ and C is $n \times k$.

As an example of a model, let us consider the simple macro model

$$\begin{aligned}
&E_t y_{t+1} + \theta E_t \pi_{t+1} = y_t + \theta R_t \\
&\beta E_t \pi_{t+1} = \pi_t - \alpha y_t \\
&R_t = \psi \pi_t + g_t \\
&g_t = \rho g_{t-1} + \varepsilon_t
\end{aligned}$$

Let us then recall that by definition of an expectation error, we have

$$\begin{aligned}
\pi_t &= E_{t-1}\pi_t + \eta_t^\pi \\
y_t &= E_{t-1} y_t + \eta_t^y
\end{aligned}$$

Plugging the definition of R_t into the first two equations, and making use of the definition of expectation errors, the system rewrites

$$\begin{aligned}
&y_t = E_{t-1} y_t + \eta_t^y \\
&\pi_t = E_{t-1}\pi_t + \eta_t^\pi \\
&E_t y_{t+1} + \theta E_t \pi_{t+1} - y_t - \theta\psi\pi_t - \theta g_t = 0 \\
&\beta E_t \pi_{t+1} - \pi_t + \alpha y_t = 0 \\
&g_t = \rho g_{t-1} + \varepsilon_t
\end{aligned}$$

Now defining $Y_t = (y_t, \pi_t, E_t y_{t+1}, E_t \pi_{t+1}, g_t)'$, the system may be writte[10]

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & -\theta\psi & 1 & \theta & 1 \\ \alpha & -1 & 0 & \beta & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} Y_t = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho \end{pmatrix} Y_{t-1} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \varepsilon_t + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \eta_t^y \\ \eta_t^\pi \end{pmatrix}$$

A nice feature of this representation is that it makes full use of expectation errors and therefore may be given a fully interpretable economic meaning.

1.5.3 Solving the system

We now turn to the resolution of the system (1.19). Since, A_0 is not necessarily invertible, we will make full use of the generalized Schur decomposition of (A_0, A_1). There therefore exist matrices Q, Z, T and S such that

$$Q'TZ' = A_0,\ Q'SZ' = A_1, QQ' = ZZ' = I_{n\times n}$$

[10] Note that $Y_{t-1} = (y_{t-1}, \pi_{t-1}, E_{t-1} y_t, E_{t-1}\pi_t, g_{t-1})$

and T and S are upper triangular. Let us then define $X_t = Z'Y_t$ and pre–multiply (1.19) by Q to get

$$\begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix} \begin{pmatrix} W_{1,t} \\ W_{2,t} \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{pmatrix} \begin{pmatrix} W_{1,t-1} \\ W_{2,t-1} \end{pmatrix} + \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} (B\varepsilon_t + C\eta_t) \tag{1.20}$$

Let us assume, without loss of generality that the system is ordered and partitioned such that the $m \times 1$ vector $W_{2,t}$ is purely explosive. Accordingly, the remaining $n - m \times 1$vector $W_{1,t}$ is stable. Let us first focus on the explosive part of the system

$$T_{22}W_{2,t} = S_{22}W_{2,t-1} + Q_2(B\varepsilon_t + C\eta_t)$$

For this particular block, the diagonal elements of T_{22} can be null, while S_{22} is necessarily full rank, as its diagonal elements must be different from zero if the model is not degenerate. Therefore, the model may be written

$$W_{2,t} = MW_{2,t+1} - S_{22}^{-1}Q_2(B\varepsilon_{t+1} + C\eta_{t+1}) \text{ where } M \equiv S_{22}^{-1}T_{22}$$

Iterating forward, we get

$$W_{2,t} = \lim_{t \longrightarrow \infty} M^s W_{2,t+s} - \sum_{s=1}^{\infty} M^{s-1} S_{22}^{-1} Q_2(B\varepsilon_{t+s} + C\eta_{t+s})$$

In order to get rid of bubbles, we have to impose $\lim_{t\longrightarrow\infty} M^s W_{2,t+s} = 0$, such that

$$W_{2,t} = -\sum_{s=1}^{\infty} M^{s-1} S_{22}^{-1} Q_2(B\varepsilon_{t+s} + C\eta_{t+s})$$

Note that by definition of the vector Y_t which does not involve any variable which do not belong to the information set available in t, we should have $E_t W_{2,t} = W_{2,t}$. But,

$$E_t W_{2,t} = -E_t \sum_{s=1}^{\infty} M^{s-1} S_{22}^{-1} Q_2(B\varepsilon_{t+s} + C\eta_{t+s}) = 0$$

This therefore imposes a restriction on ε_t and η_t. Indeed, if we go back to the recursive formulation of $W_{2,t}$ and take into account that $W_{2,t} = 0$ for all t, this imposes

$$\underbrace{Q_2 B}_{(m \times \ell)} \quad \underbrace{\varepsilon_t}_{(\ell \times 1)} \quad + \quad \underbrace{Q_2 C}_{(m \times k)} \quad \underbrace{\eta_t}_{(k \times 1)} \quad = \quad \underbrace{0}_{(m \times 1)} \tag{1.21}$$

Our problem is now to know whether we can pin down the vector of expectation errors uniquely from that set of restrictions. Indeed, the vector η_t may not be uniquely determined. This is the case for instance when the number of expectation errors k exceeds the number of explosive components m. In this case, equation (1.21) does not provide enough restrictions to determine uniquely the vector η_t. In other words, it is possible to introduce expectation errors which are not related with fundamental uncertainty — the so–called sunspot variables.

Sims [2000] shows that a necessary and sufficient condition for a stable solution to exist is that the column space of Q_2B be contained in the column space of Q_2C:

$$\text{span}(Q_2B) \subset \text{span}(Q_2C)$$

Otherwise stated, we can reexpress Q_2B as a linear function of Q_2C ($Q_2B = Q_2C\Theta$), implying that $k \geqslant m$. This is actually a generalization of the so–called Blanchard and Khan condition that states that the number of explosive eigenvalues should be equal to the number of jump variables in the system. Lubik and Schorfheide [2003] complement this statement by the following lemma.

Lemma 1 *Statements (i) and (ii) are equivalent*

(i) For every $\varepsilon_t \in \mathbb{R}^\ell$, there exists an $\eta_t \in \mathbb{R}^k$ such that $Q_2B\varepsilon_t + Q_2C\eta_t = 0$.

(ii) There exists a (real) $k \times \ell$ matrix Θ such that $Q_2B = Q_2C\Theta$

Endowed with this lemma, we can compute the set of all solutions (fully determinate and indeterminate solutions), reported in the following proposition.

Proposition 5 (Lubik and Schorfheide [2003]) *Let ξ_t be a $p \times 1$ vector of sunspot shocks, satisfying $E_{t-1}\xi_t = 0$. Suppose that condition (i) of lemma 1 is satisfied. The full set of solutions for the forecast errors in the linear rational expectations model is*

$$\eta_t = (-V_1 D_{11}^{-1} U_1' Q_2 B + V_2 M_1)\varepsilon_t + V_2 M_2 \xi_t$$

where M_1 is a $(k-r) \times \ell$ matrix and M_2 is a $(k-r) \times p$ matrix.

Proof: First of all, we have to find a solution to equation (1.21). The problem is that the rows of matrix Q_2C can be linearly dependent. Therefore, we will use the *Singular Value Decomposition* of Q_2C

$$Q_2C = \underbrace{U}_{m\times m}\ \underbrace{D}_{m\times k}\ \underbrace{V'}_{k\times k}$$

which may be partitioned as

$$Q_2C = (U_1\ U_2)\begin{pmatrix} D_{11} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} V_1' \\ V_2' \end{pmatrix} = U_1D_{11}V_1'$$

where D_{11} is a $r\times r$ matrix, where r is the number of linearly independent rows in Q_2C — therefore the actual number of restrictions. Accordingly, U_1 is $m\times r$, and V_1 is $k\times r$.

Given that we are looking for a solution that satisfies $Q_2B = Q_2C\Theta$, equation (1.21) rewrites

$$\underbrace{U_1D_{11}}_{m\times r}\ \underbrace{(V_1'\Theta\varepsilon_t + V_1'\eta_t)}_{r\times 1} = \underbrace{0}_{m\times 1}$$

We therefore now have r restrictions to identify the k–dimensional vector of expectation errors.

We guess that the solution implies that forecast errors are a linear function of *(i)* fundamental shocks and *(ii)* a $p\times 1$ vector of sunspot shocks ξ_t, satisfying $E_{t-1}\xi_t = 0$:

$$\eta_t = \Gamma_\varepsilon\varepsilon_t + \Gamma_\xi\xi_t$$

where Γ_ε is $k\times\ell$ and Γ_ξ is $k\times p$.

Plugging this guess in the former equation, we get

$$U_1D_{11}(V_1'\Theta + V_1'\Gamma_\varepsilon)\varepsilon_t + U_1D_{11}V_1'\Gamma_\xi\xi_t = 0$$

for all ε_t and ξ_t. This triggers that we should have

$$U_1D_{11}(V_1'\Theta + V_1'\Gamma_\varepsilon) = 0 \tag{1.22}$$

$$U_1D_{11}V_1'\Gamma_\xi = 0 \tag{1.23}$$

Let us first focus on equation (1.22). Since V is an orthonormal matrix, it satisfies $VV' = I$ — otherwise stated $V_1V_1' + V_2V_2' = I$ — and $V'V = I$, implying that $V_1'V_2 = 0$. A direct consequence of the first part of this statement is that

$$\Gamma_\varepsilon = V_1(V_1'\Gamma_\varepsilon) + V_2(V_2'\Gamma_\varepsilon) = V_1\widetilde{\Gamma}_\varepsilon + V_2M_1$$

with $\widetilde{\Gamma}_\varepsilon \equiv V_1'\Gamma_\varepsilon$ and $M_1 \equiv V_2'\Gamma_\varepsilon$. Since $V_1'V_1 = I$ and $V_1'V_2 = 0$, (1.22) therefore rewrites

$$U_1D_{11}(V_1'\Theta + \widetilde{\Gamma}_\varepsilon) = 0$$

from which we get

$$\widetilde{\Gamma}_\varepsilon = -V_1'\Theta$$

We still need to identify Θ to determine $\widetilde{\Gamma}_\varepsilon$ and therefore Γ_ε. To do so, we use the fact that $Q_2B = Q_2C\Theta$ and $Q_2C = U_1D_{11}V_1'$, to get

$$Q_2B = Q_2C\Theta = U_1D_{11}V_1'\Theta$$

Since U is orthonormal, we have $U_1'U_1 = I$, such that

$$V_1'\Theta = U_1'D_{11}^{-1}Q_2B$$

Therefore, plugging this result in the determination of $\widetilde{\Gamma}_\varepsilon$, we get

$$\widetilde{\Gamma}_\varepsilon = -D_{11}^{-1}U_1'Q_2B$$

Since $\Gamma_\varepsilon = V_1\widetilde{\Gamma}_\varepsilon + V_2M_1$, we finally get

$$\Gamma_\varepsilon = -V_1D_{11}^{-1}U_1'Q_2B + V_2M_1$$

where M_1 is left totally undetermined and therefore arbitrary.

We can now focus on (1.23) to determine Γ_ξ. This is actually straightforward as it simply triggers that Γ_ξ be orthogonal to V_1. But since $V_1V_2' = 0$, the orthogonal space of V_1 is spanned by the columns of the $k \times (k-r)$ matrix V_2. In other words, any linear combination of the column of V_2 would do the job. Hence

$$\Gamma_\xi = V_2M_2$$

where once again M_2 is left totally undetermined and therefore arbitrary.

□

This last result tells us how to solve the model and under which condition the system is determined or not. Indeed, let us recall that k is the number of expectation errors, while r is the number of linearly independent expectation errors. According to this proposition, if $k = r$, all expectation errors are linearly independent, and the system is therefore totally determinate. M_1 and M_2 are identically zeros. Conversely, if $k > r$ expectation errors are not linearly independent, meaning that the system does not provide enough restrictions to uniquely pin down the expectation errors. We therefore have to introduce extrinsic uncertainty in the system — the so–called sunspot variables. We will deal first with the determinate case, before considering the case of indeterminate system.

Determinacy

This case occurs when the number of expectation errors exactly matches the number of explosive components ($k = m$), or otherwise stated in the case where $k = r$. As shown in proposition 5, the expectation errors are then just a linear combination of fundamental disturbances for all t since both M_1 and M_2 reduce to nil matrices. Therefore, in this case, we have

$$\eta_t = -V_1 D_{11}^{-1} U_1' Q_2 B \varepsilon_t$$

such that the overall effect of fundamental shocks on W_t is

$$(Q_1 B - Q_1 C V_1 D_{11}^{-1} U_1' Q_2 B)\varepsilon_t$$

while that of purely extrinsic expectation errors is nil. To get such an effect in the first part of system (1.20), we shall pre–multiply by the matrix $[I \vdots -\Phi]$ where $\Phi \equiv Q_1 C V_1 D_{11}^{-1} U_1'$. Then, taking into account that $W_{2t} = 0$, we have

$$\begin{aligned}\begin{pmatrix} T_{11} & T_{12} - \Phi T_{22} \\ 0 & I \end{pmatrix}\begin{pmatrix} W_{1,t} \\ W_{2,t} \end{pmatrix} &= \begin{pmatrix} S_{11} & S_{12} - \Phi S_{22} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} W_{1,t-1} \\ W_{2,t-1} \end{pmatrix} \\ &+ \begin{pmatrix} Q_1 - \Phi Q_2 \\ 0 \end{pmatrix} B\varepsilon_t\end{aligned}$$

Noting that the inverse of the matrix

$$\begin{pmatrix} T_{11} & T_{12} - \Phi T_{22} \\ 0 & I \end{pmatrix}$$

is

$$\begin{pmatrix} T_{11}^{-1} & -T_{11}^{-1}(T_{12} - \Phi T_{22}) \\ 0 & I \end{pmatrix}$$

we have

$$\begin{pmatrix} W_{1,t} \\ W_{2,t} \end{pmatrix} = \begin{pmatrix} T_{11}^{-1} S_{11} & T_{11}^{-1}(S_{12} - \Phi S_{22}) \\ 0 & 0 \end{pmatrix}\begin{pmatrix} W_{1,t-1} \\ W_{2,t-1} \end{pmatrix} + \begin{pmatrix} T_{11}^{-1}(Q_1 - \Phi Q_2) \\ 0 \end{pmatrix} B\varepsilon_t$$

Now recall that $W_t = Z'Y_t$ and that $ZZ' = I$. Therefore, pre–multiplying the last equation by Z, we end up with a solution of the form

$$Y_t = M_{\text{Y}} Y_{t-1} + M_{\text{E}} \varepsilon_t \tag{1.24}$$

with

$$M = Z\begin{pmatrix} T_{11}^{-1} S_{11} & T_{11}^{-1}(S_{12} - \Phi S_{22}) \\ 0 & 0 \end{pmatrix} Z' \text{ and } M_{\text{E}} = Z\begin{pmatrix} T_{11}^{-1}(Q_1 - \Phi Q_2) \\ 0 \end{pmatrix} B$$

Indeterminacy

This case arises as soon as the number of expectation errors is greater than the number of explosive components ($k > m$), which translates into the fact that $k > r$. As shown in proposition 5, the expectation errors are then not only linear combinations of fundamental disturbances for all t but also of purely extrinsic disturbances called sunspot variables. Then, the expectation errors are shown to be of the form

$$\eta_t = (-V_1 D_{11}^{-1} U_1' Q_2 B + V_2 M_1)\varepsilon_t + V_2 M_2 \xi_t$$

where both M_1 and M_2 can be freely chosen. This actually raises several questions. The first one is *how to select* M_1 *and* M_2*?* They are totally arbitrary the only restriction we have to impose is that M_1 is a $(k-r)\times\ell$ matrix and M_2 is a $(k-r)\times p$ matrix. A second one is then *how to interpret these sunspots?* In order to partially circumvent these difficulties, it is useful to introduce the notion of beliefs. For instance, this amounts to introduce new shocks — the sunspots — beside the standard expectation error. In such a case, a variable y_t will be determined by its expectation at time $t-1$, a shock on the beliefs that leads to a revision of forecasts, and the expectation error

$$y_t = E_{t-1} y_t + \zeta_t + \overline{\eta}_t$$

where ζ_t is the shock on the belief, that satisfies $E_{t-1}\zeta_t = 0$, and $\overline{\eta}_t$ is the expectation error. ζ_t is a $k \times 1$ vector. Then the system 1.19 rewrites

$$A_0 Y_t = A_1 Y_{t-1} + B\varepsilon_t + C(\zeta_t + \overline{\eta}_t)$$

which can be restated in the form

$$A_0 Y_t = A_1 Y_{t-1} + \overline{B}\begin{pmatrix} \varepsilon_t \\ \zeta_t \end{pmatrix} + C\overline{\eta}_t$$

where $\overline{B} = [B\ C]$. Implicit in this rewriting of the system is the fact that the belief shock be treated like a fundamental shock, therefore condition (1.21) rewrites

$$Q_2\overline{B}\begin{pmatrix} \varepsilon_t \\ \zeta_t \end{pmatrix} + Q_2 C\overline{\eta}_t = 0$$

which leads, according to proposition 5, to an expectation error of the form

$$\overline{\eta_t} = (-V_1 D_{11}^{-1} U_1' Q_2 B + V_2 M_1^{\varepsilon})\varepsilon_t + (-V_1 D_{11}^{-1} U_1' Q_2 C + V_2 M_1^{\zeta})\zeta_t$$

But, since $Q_2C = U_1D_{11}V_1'$ and $V_1V_1' + V_2V_2' = I$, this rewrites

$$\overline{\eta}_t = (-V_1D_{11}^{-1}U_1'Q_2B + V_2M_1^{\varepsilon})\varepsilon_t + V_2(V_2' + M_1^{\zeta})\zeta_t$$

This shows that the expectation error is a function of both the fundamental shocks and the beliefs.
If this latter formulation furnishes an economic interpretation to the sunspots, it leaves unidentified the matrices M_1^{ε} and M_1^{ζ}. From a practical point of view, we can, arbitrarily, set these matrices to zeros and then proceed exactly as in the determinate case, replacing B by $\overline{B}$ in the solution. This leads to

$$Y_t = M_{\text{Y}}Y_{t-1} + M_{\text{E}}\begin{pmatrix} \varepsilon_t \\ \zeta_t \end{pmatrix} \tag{1.25}$$

with

$$M = Z\begin{pmatrix} T_{11}^{-1}S_{11} & T_{11}^{-1}(S_{12} - \Phi S_{22}) \\ 0 & 0 \end{pmatrix}Z' \text{ and } M_{\text{E}} = Z\begin{pmatrix} T_{11}^{-1}(Q_1 - \Phi Q_2) \\ 0 \end{pmatrix}\overline{B}$$

Note however, that even if we know the form of the solution, we know nothing about the statistical properties of the ζ_t shocks. In particular, we do not know their covariance matrix that can be set arbitrarily.

1.5.4 Using the model

In this section, we will show you how the solution may be used to study the dynamic properties of the model from a quantitative point of view. We will basically address two issues

1. Impulse response functions
2. Computation of moments

Impulse response functions

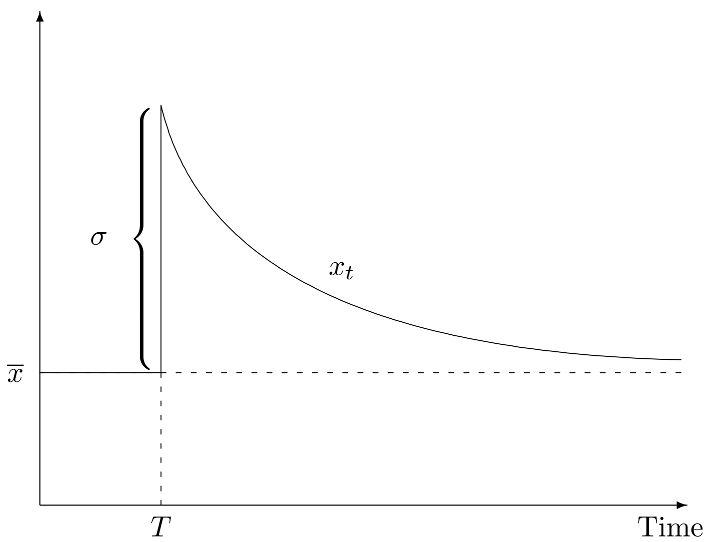
As we have already seen in the preceding chapter, the *impulse response function* of a variable to a shock gives us the expected response of the variable to a shock at different horizons — in other words this corresponds to the best linear predictor of the variable if the economic environment remains the same in the future. For instance, and just to remind you what it is, let us consider the case of an AR(1) process:

$$x_t = \rho x_{t-1} + (1-\rho)\overline{x} + \varepsilon_t$$

Assume for a while that no shocks occurred in the past, such that x_t remained steady at the level $\overline{x}$ from $t = 0$ to T. A unit positive shock of magnitude σ occurs in T, x_T is then given by

$$x_T = \overline{x} + \sigma$$

Figure 1.12: Impulse Response Function (AR(1))



In $T + 1$, no other shock feeds the process, such that x_{T+1} is given by

$$x_{T+1} = \rho x_T + (1 - \rho)\overline{x} = \overline{x} + \rho\sigma$$

X_{T+2} is then given by

$$x_{T+2} = \rho x_{T+1} + (1 - \rho)\overline{x} = \overline{x} + \rho^2\sigma$$

therefore, as reported in figure 1.12, we have

$$x_{T+i} = \rho x_{T+i-1} + (1 - \rho)\overline{x} = \overline{x} + \rho^i\sigma \quad \forall i \geqslant 1$$

In our system, obtaining impulse response functions is as simple as that, provided the solution has already been computed. Assume we want to compute the response to one of the fundamental shocks ($\varepsilon_{i,t} \in \mathcal{E}_t$). On impact the vector of endogenous variables (Y_t)responds as

$$Y_t = M_E \times e_i \text{ with } e_{ik} = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{otherwise} \end{cases}$$

The response as horizon j is then given by:

$$Y_{t+j} = M_{\mathrm{Y}} Y_{t+j-1} \quad j > 0$$

Computation of moments

Let us focus on the computation of the moments for this economy. We will describe two ways to do it. The first one uses a direct theoretical computation of the moments, while the second one relies on Monte–Carlo simulations.

The theoretical computation of moments can be achieved in a straightforward way. Let us focus for a while on the covariance matrix of the state variables:

$$\Sigma_{\mathrm{YY}} = E(Y_t Y_t')$$

Recall that in the most complicated case, we have

$$Y_t = M_{\mathrm{Y}} Y_{t-1} + M_{\mathrm{E}} \varepsilon_t$$

with $E(\varepsilon_t \varepsilon_t') = \Sigma_{\mathrm{EE}}$.
Further, recall that we only consider stationary representations of the economy, such that $\Sigma_{SS} = E(S_{t+j} S_{t+j}')$ whatever j. Hence, we have

$$\Sigma_{\mathrm{YY}} = M_{\mathrm{Y}} \Sigma_{\mathrm{YY}} M_{\mathrm{Y}}' + M_{\mathrm{Y}} E(Y_{t-1} \varepsilon_t') M_{\mathrm{E}} + + M_{\mathrm{E}} E(\varepsilon_t Y_{t-1}') M_{\mathrm{Y}}' + M_{\mathrm{E}} \Sigma_{\mathrm{EE}} M_{\mathrm{E}}'$$

Since both ε_t are innovations, they are orthogonal to Y_t, such that the previous equation reduces to

$$\Sigma_{\mathrm{YY}} = M_{\mathrm{Y}} \Sigma_{\mathrm{YY}} M_{\mathrm{Y}}' + M_{\mathrm{E}} \Sigma_{\mathrm{EE}} M_{\mathrm{E}}'$$

Solving this equation for Σ_{SS} can be achieved remembering that $\mathrm{vec}(ABC) = (A \otimes C')\mathrm{vec}(B)$, hence

$$\mathrm{vec}(\Sigma_{\mathrm{YY}}) = (I - M_{\mathrm{Y}} \otimes M_{\mathrm{Y}})^{-1} \mathrm{vec}(\Sigma_{\mathrm{EE}})$$

The computation of covariances at leads and lags proceeds the same way. For instance, assume we want to compute $\Sigma_{SS}^j = E(S_t S'_{t-j})$. From

$$Y_t = M_{\mathrm{Y}} Y_{t-1} + M_{\mathrm{E}} \varepsilon_t$$

we know that

$$Y_t = M_{\mathrm{Y}}^j Y_{t-j} + M_{\mathrm{E}} \sum_{i=0}^{j} M_{\mathrm{Y}}^i \varepsilon_{t-i}$$

Therefore,

$$E(Y_t Y'_{t-j}) = M_{\mathrm{Y}}^j E(Y_{t-j} Y'_{t-j}) + M_{\mathrm{E}} \sum_{i=0}^{j} M_{\mathrm{Y}}^i E(\varepsilon_{t-i} Y'_{t-j})$$

Since ε are innovations, they are orthogonal to any past value of Y, such that

$$E(\varepsilon_{t-i} Y'_{t-j}) = \begin{cases} 0 & \text{if } i < j \\ \Sigma_{\mathrm{EE}} M'_{\mathrm{E}} & \text{if } i = j \end{cases}$$

Then, the previous equation reduces to

$$E(Y_t Y'_{t-j}) = M_{\mathrm{Y}}^j \Sigma_{\mathrm{YY}} + M_{\mathrm{E}} M_{\mathrm{Y}}^j \Sigma_{\mathrm{EE}} M'_{\mathrm{E}}$$

The Monte–Carlo simulation is as simple as computing Impulse Response Functions, as it just amounts to simulate a process for ε, impose an initial condition for Y_0 and to iterate on

$$Y_t = M_{\mathrm{Y}} Y_{t-1} + M_{\mathrm{E}} \varepsilon_t \text{ for } t = 0, \ldots, T$$

Then moments can be computed and stored in a matrix. The experiment is conducted N times, as $N \longrightarrow \infty$ one can compute the asymptotic distribution of the moments.

1.6 Economic examples

This section intends to provide you with some economic applications of the set of tools we have described up to now. We will consider three examples, two of which may be thought of as micro examples. In the first one a firm decides on its labor demand, the second one is a macro model — and endogenous growth model à la Romer [1986] — which allows to show that even a non–linear model may be expressed in linear terms and therefore may be solved in a very simple way. The last one deals with the so–called Lucas critique which has strong implications on the econometric side.

1.6.1 Labor demand

We consider the case of a firm that has to decide on its level of employment. The firm is infinitely lived and produces a good relying on a decreasing returns to scale technology that essentially uses labor — another way to think of it would be to assume that physical capital is a fixed–factor. This technology is represented by the production function

$$Y_t = f_0 n_t - \frac{f_1}{2} n_t^2 \text{ with } f_0, f_1 > 0.$$

Using labor incurs two sources of cost

1. The standard payment for labor services: $w_t n_t$ where w_t is the real wage, which positive sequence $\{w_t\}_{t=0}^{\infty}$ is taken as given by the firm

2. A cost of adjusting labor which may be justified either by appealing to reorganization costs, training costs, and that takes the form

$$\frac{\varphi}{2}(n_t - n_{t-1})^2 \text{ with } \varphi > 0$$

Labor is then determined by maximizing the expected intertemporal profit

$$\max_{\{n_\tau\}_{\tau=0}^{\infty}} E_t \sum_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^s \left(f_0 n_{t+s} - \frac{f_1}{2} n_{t+s}^2 - w_{t+s} n_{t+s} - \frac{\varphi}{2}(n_{t+s} - n_{t+s-1})^2\right)$$

First order conditions: Finding the first order conditions associated to this dynamic optimization problem may be achieved in various ways. Here, we will follow Sargent [1987] and will adopt the Lagrangean approach. Let us fix s for a while and make some accountancy in order to find all the terms involving n_{t+s}

in $s-i, i=2,\dots$	none
in $s-1$	none
in s	$E_t \left(\frac{1}{1+r}\right)^s \left(f_0 n_{t+s} - \frac{f_1}{2} n_{t+s}^2 - w_{t+s} n_{t+s} - \frac{\varphi}{2}(n_{t+s} - n_{t+s-1})^2\right)$
in $s+1$	$E_t \left(\frac{1}{1+r}\right)^{s+1} \left(-\frac{\varphi}{2}(n_{t+s+1} - n_{t+s})^2\right)$
in $s+i, i=2,\dots$	none

Hence, finding the optimality condition associated to n_{t+s} reduces to maximizing

$$E_t\left(\frac{1}{1+r}\right)^s\left[f_0 n_{t+s}-\frac{f_1}{2}n_{t+s}^2-w_{t+s}n_{t+s}-\frac{\varphi}{2}(n_{t+s}-n_{t+s-1})^2-\left(\frac{1}{1+r}\right)\frac{\varphi}{2}(n_{t+s+1}-n_{t+s})^2\right]$$

which yields the following first order condition

$$E_t\left(\frac{1}{1+r}\right)^s\left[f_0-f_1 n_{t+s}-w_{t+s}-\varphi(n_{t+s}-n_{t+s-1})+\left(\frac{1}{1+r}\right)\varphi(n_{t+s+1}-n_{t+s})\right]=0$$

since r is a constant this reduces to

$$E_t\left[f_0-f_1 n_{t+s}-w_{t+s}-\varphi(n_{t+s}-n_{t+s-1})+\left(\frac{1}{1+r}\right)\varphi(n_{t+s+1}-n_{t+s})\right]=0$$

Now remark that this relationship holds whatever s, such that we may restrict ourselves to the case $s=0$ which then yields — noting that n_{t-i}, $i\geqslant 0$ belongs to the information set

$$f_0-f_1 n_t-w_t-\varphi(n_t-n_{t-1})+\frac{\varphi}{1+r}(E_t n_{t+1}-n_t)=0$$

rearranging terms

$$E_t n_{t+1}-\left(2+r+\frac{f_1(1+r)}{\varphi}\right)n_t+(1+r)n_{t-1}+\frac{1+r}{\varphi}(f_0-w_t)=0$$

Finally we have the transversality condition

$$\lim_{T\to+\infty}(1+r)^{-T}\varphi(n_T-n_{T-1})n_T=0$$

Solving the model: In this example, we will apply all three methods that we have described previously. Let us first start with factorization.

The preceding equation may be rewritten using the forward operator as

$$P(F)n_{t-1}\equiv\left(F^2-\left(2+r+\frac{f_1(1+r)}{\varphi}\right)F+1+r\right)n_{t-1}=\frac{1+r}{\varphi}(w_t-f0)$$

$P(F)$ may be factorized as

$$P(F)=(F-\mu_1)(F-\mu_2)$$

Let us compute the discriminant of this second order polynomial

$$\Delta\equiv\left(2+r+\frac{f_1(1+r)}{\varphi}\right)^2-4(1+r)=(1+r)\frac{f_1}{\varphi}\left((1+r)\frac{f_1}{\varphi}+2(2+r)\right)>0$$

Hence, since $\Delta > 0$, we know that the two roots are real. Further

$$\begin{aligned} P(1) &= -\frac{f_1(1+r)}{\varphi} < 0 \\ P(-1) &= (1+r)\frac{f_1}{\varphi} + 2(2+r) > 0 \\ P(0) &= 1+r > 0 \\ P'(x) &= 0 \Longleftrightarrow x = \frac{1}{2}\left(2+r+\frac{f_1(1+r)}{\varphi}\right) > 1 \end{aligned}$$

$P(0)$ being greater than 0 and since P(1) is negative, one root lies between 0 and 1, and the other one is therefore greater than 1 since $\lim_{x\longrightarrow\infty} P(x) = \infty$. The system therefore satisfies the saddle path property.

Let us assume then that $\mu_1 < 1$ and $\mu_2 > 1$. The expectational equation rewrites

$$(F-\mu_1)(F-\mu_2)n_{t-1} = w_t - f_0 \Longleftrightarrow (F-\mu_1)n_{t-1} = \frac{1+r}{\varphi}\frac{w_t - f0}{F-\mu_2}$$

or

$$n_t = \mu_1 n_{t-1} + \frac{1+r}{\mu_2\varphi}\frac{f_0 - w_t}{1-\mu_2^{-1}F} = \mu_1 n_{t-1} + \frac{1+r}{\mu_2\varphi}\sum_{i=0}^{\infty}\mu_2^{-i}E_t(f_0 - w_{t+i})$$

Since $\mu_1\mu_2 = (1+r)$, this rewrites

$$n_t = \mu_1 n_{t-1} + \frac{\mu_1}{\varphi}\sum_{i=0}^{\infty}\mu_2^{-i}E_t(f_0 - w_{t+i})$$

or developing the series

$$n_t = \frac{f_0(1+r)}{\varphi(\mu_2-1)} + \mu_1 n_{t-1} - \frac{\mu_1}{\varphi}\sum_{i=0}^{\infty}\mu_2^{-i}E_t w_{t+i}$$

For practical purposes let us assume that w_t follows an AR(1) process of the form

$$w_t = \rho w_{t-1} + (1-\rho)\overline{w} + \varepsilon_t$$

we have

$$E_t w_{t+i} = \rho^i w_t + (1-\rho^i)\overline{w}$$

such that n_t rewrites

$$n_t = \frac{f_0(1+r)}{\varphi(\mu_2-1)} - \frac{(1+r)(1-\rho)}{\varphi(\mu_2-1)(\mu_2-\rho)}\overline{w} + \mu_1 n_{t-1} - \frac{1+r}{\varphi(\rho-\mu_2)}w_t$$

We now consider the problem from the method of undetermined coefficients point of view, and guess that the solution takes the form

$$n_t = \alpha_0 + \alpha_1 n_{t-1} + \sum_{i=0}^{\infty} \gamma_i E_t w_{t+i}$$

Plugging the guess in $t+1$ in the Euler equation, we get

$$\begin{aligned} & E_t \left(\alpha_0 + \alpha_1 \left(\alpha_0 + \alpha_1 n_{t-1} + \sum_{i=0}^{\infty} \gamma_i E_t w_{t+i} \right) + \sum_{i=0}^{\infty} \gamma_i E_{t+1} w_{t+i+1} \right) \\ & - \left(2 + r + \frac{f_1(1+r)}{\varphi} \right) \left(\alpha_0 + \alpha_1 n_{t-1} + \sum_{i=0}^{\infty} \gamma_i E_t w_{t+i} \right) \\ & + (1+r) n_{t-1} + \frac{1+r}{\varphi}(f_0 - w_t) = 0 \end{aligned}$$

which rewrites

$$\begin{aligned} & \alpha_0(1+\alpha_1) + \alpha_1^2 n_{t-1} + \alpha_1 \sum_{i=0}^{\infty} \gamma_i E_t w_{t+i} + \sum_{i=0}^{\infty} \gamma_i E_t w_{t+i+1} \\ & - \left(2 + r + \frac{f_1(1+r)}{\varphi} \right) \left(\alpha_0 + \alpha_1 n_{t-1} + \sum_{i=0}^{\infty} \gamma_i E_t w_{t+i} \right) \\ & + (1+r) n_{t-1} + \frac{1+r}{\varphi}(f_0 - w_t) = 0 \end{aligned}$$

Identifying term by term, we get the system

$$\left\{ \begin{array}{l} \alpha_0(1+\alpha_1) - \left(2 + r + \frac{f_1(1+r)}{\varphi} \right) \alpha_0 + \frac{1+r}{\varphi} f_0 = 0 \\ \alpha_1^2 - \left(2 + r + \frac{f_1(1+r)}{\varphi} \right) \alpha_1 + (1+r) = 0 \\ \gamma_0 \left(\alpha_1 - \left(2 + r + \frac{f_1(1+r)}{\varphi} \right) \right) - \frac{1+r}{\varphi} = 0 \\ \gamma_i \left(\alpha_1 - \left(2 + r + \frac{f_1(1+r)}{\varphi} \right) \right) + \gamma_{i-1} = 0 \end{array} \right.$$

The second equation of the system exactly corresponds to the second order polynomial we solved in the factorization method. The system therefore exhibits the saddle path property so that $\mu_1 \in (0,1)$ and $\mu_2 \in (1,\infty)$. Let us recall that $\mu_1 + \mu_2 = 2 + r + f_1(1+r)/\varphi$, such that the system for α_0 and γ_i rewrites

$$\left\{ \begin{array}{l} \alpha_0(1+\alpha_1) - \left(2 + r + \frac{f_1(1+r)}{\varphi} \right) \alpha_0 + \frac{1+r}{\varphi} f_0 = 0 \\ -\gamma_0 \mu_2 - \frac{1+r}{\varphi} = 0 \\ \gamma_i = \mu_2^{-1} \gamma_{i-1} \end{array} \right.$$

Therefore, we have

$$\gamma_0 = -\frac{1+r}{\varphi\mu_2} = -\frac{\mu_1}{\varphi}$$

and $\gamma_i = \mu_2^{-i}\gamma_0$. Finally, we have

$$\alpha_0 = \frac{f_0(1+r)}{\varphi(\mu_2-1)}$$

We then find the previous solution

$$n_t = \frac{f_0(1+r)}{\varphi(\mu_2-1)} + \mu_1 n_{t-1} - \frac{\mu_1}{\varphi}\sum_{i=0}^{\infty}\mu_2^{-i}E_t w_{t+i}$$

As a final "exercise", let us adopt the matricial approach to the problem. To do so, and because this approach is essentially numerical, we need to assume a particular process for the real wage. We will assume that it takes the preceding AR(1) form. Further, we do not need to deal with levels in this approach such that we will express the model in terms of deviation from its steady state. We thus first compute this quantity, which is defined by

$$n^\star - \left(2+r+\frac{f_1(1+r)}{\varphi}\right)n^\star + (1+r)n^\star + \frac{1+r}{\varphi}(f_0-\overline{w}) = 0 \Longleftrightarrow n^\star = \frac{f_0-\overline{w}}{f_1}$$

Denoting $\widehat{n}_t = n_t - n^\star$ and $\widehat{w}_t = w_t - \overline{w}$, and introducing the "technical variable" $\widehat{z}_{t+1} = \widehat{n}_t$, the Labor demand re–expresses as

$$E_t\widehat{n}_{t+1} - \left(2+r+\frac{f_1(1+r)}{\varphi}\right)\widehat{n}_t + (1+r)\widehat{z}_t - \frac{1+r}{\varphi}\widehat{w}_t = 0$$

We define the vector $Y_t = \{\widehat{z}_{t+1}, \widehat{n}_t, \widehat{w}_t, E_t\widehat{n}_{t+1}\}$. Remembering that $\widehat{n}_t = E_{t-1}\widehat{n}_t + \eta_t$, the system expresses as

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -\left(2+r+\frac{f_1(1+r)}{\varphi}\right) & -\frac{1+r}{\varphi} & 0 \end{pmatrix}\begin{pmatrix} \widehat{z}_{t+1} \\ \widehat{n}_t \\ \widehat{w}_t \\ E_t\widehat{n}_{t+1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \rho & 0 \\ -(1+r) & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \widehat{z}_t \\ \widehat{n}_{t-1} \\ \widehat{w}_{t-1} \\ E_{t-1}\widehat{n}_t \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}\varepsilon_t + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}\eta_t$$

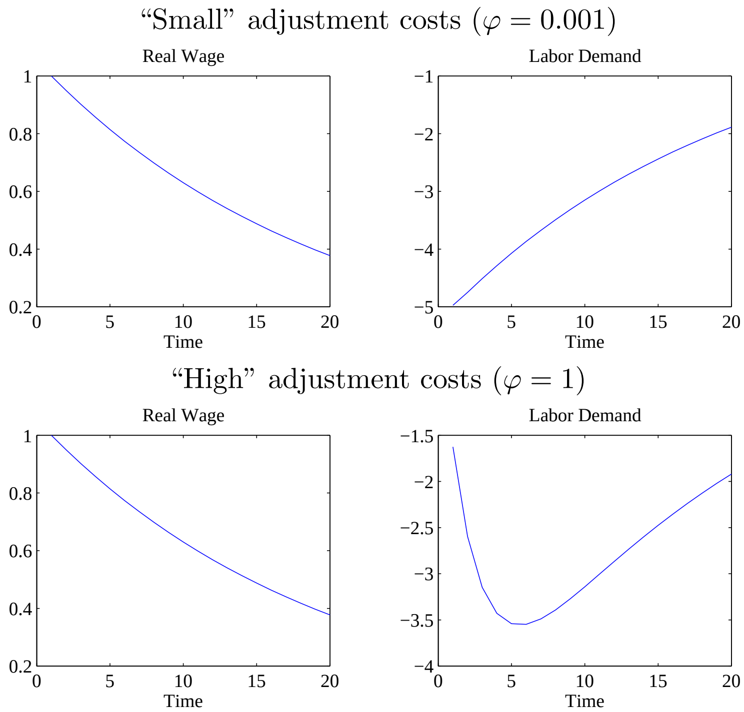
We now provide you with an example of the type of dynamics this model may generate. Figure 1.13 reports the impulse response function of labor to

Table 1.1: Parameterization: labor demand

r	f_0	f_1	φ	$\overline{w}$	ρ
0.01	1	0.2	0.001/1	0.6	0.95

a positive shock on the real wage (table 1.1 reports the parameterization). As expected, labor demand shifts downward instantaneously, but depending on the size of the adjustment cost, the magnitude of the impact effect differs. When adjustment costs are low, the firm drastically cuts employment, which goes back steadily to its initial level as the effects of the shock vanish. Conversely, when adjustment costs are high, the firm does not respond as

Figure 1.13: Impulse Response to a Wage Shock



much as before since it wants to avoid paying the cost. Nevertheless, it remains optimal to cut employment, so in order to minimize the cost, the firm spreads it intertemporally by smoothing the employment profile, therefore generating a hump shaped response of employment.

Matlab Code: Labor Demand

```
%
% Labor demand
%
clear all
%
% Structural Parameters
%
r      = 0.02;
f0     = 1;
wb     = 0.6;
f1     = 0.2;
phi    = 1;
rho    = 0.95;
nb     = (f0-wb)/f1;

A0=[
    1 -1 0 0
    0  1 0 0
    0  0 1 0
    0 -(2+r+f1*(1+r)/phi) -(1+r)/phi 1
];

A1=[
    0 0 0 0
    0 0 0 1
    0 0 rho 0
    -(1+r) 0 0 0
];

B=[0;0;1;0];
C=[0;1;0;0];
% Call Sims Routine
[MY,ME] = sims_solve(A0,A1,B,C);
%
% IRF
%
nrep   = 20;
SHOCK  = 1;
YS     = zeros(4,nrep);
YS(:,1)= ME*SHOCK;
for i  = 2:nrep;
  YS(:,i)=MY*YS(:,i-1);
end
T=1:nrep;
subplot(221);plot(T,Y(3,:));
subplot(222);plot(T,Y(2,:));
```

1.6.2 The Real Business Cycle Model

We consider an economy that consists of a large number of dynastic households and a large number of firms. Firms are producing a homogeneous final product that can be either consumed or invested by means of capital and labor services. Firms own their capital stock and hire labor supplied by the households. Households own the firms. In each and every period three perfectly competitive markets open — the markets for consumption goods, labor services, and financial capital in the form of firms' shares. Household preferences are characterized by the lifetime utility function:

$$E_t \sum_{s=0}^{\infty} \beta^s \log(c_{t+s}) - \Psi \frac{h_{t+s}^{1+\psi}}{1+\psi}$$

where $0 < \beta < 1$ is a constant discount factor, c_t is consumption in period t, h_t is the fraction of total available time devoted to productive activity in period t, $\Psi > 0$ and $\psi > 0$. We assume that there exists a central planner that determines hours, consumption and capital accumulation maximizing the household's utility function subject to the following budget constraint

$$c_t + i_t = y_t \tag{1.26}$$

where i_t is investment, and y_t is output. Investment is used to form physical capital, which accumulates in the standard form as:

$$k_{t+1} = i_t + (1-\delta)k_t \text{ with } 0 \leqslant \delta \leqslant 1 \tag{1.27}$$

where δ is the constant physical depreciation rate.

Output is produced by means of capital and labor services, relying on a constant returns to scale technology represented by the following Cobb– Douglas production function:

$$y_t = a_t k_t^{\alpha} h_t^{1-\alpha} \text{ with } 0 < \alpha < 1 \tag{1.28}$$

a_t represents a stochastic shock to technology or Solow residual, which evolves according to:

$$\log(a_t) = \rho \log(a_{t-1}) + (1-\rho)\log(\overline{a}) + \varepsilon_t \tag{1.29}$$

The unconditional mean of a_t is $\overline{a}$, $|\rho| < 1$ and ε_t is a gaussian white noise with standard deviation of σ. Therefore, the central planner solves

$$\max_{\{c_{t+s},k_{t+1+s}\}_{s=0}^{\infty}} E_t \sum_{s=0}^{\infty} \beta^s \log(c_{t+s}) - \Psi \frac{h_{t+s}^{1+\psi}}{1+\psi}$$

s.t.

$$\begin{aligned} k_{t+1} =& y_t = a_t k_t^{\alpha} h_t^{1-\alpha} - c_t + (1-\delta)k_t \\ \log(a_t) =& \rho \log(a_{t-1}) + (1-\rho)\log(\overline{a}) + \varepsilon_t \end{aligned}$$

The set of conditions characterizing the equilibrium is given by

$$\Psi h_t^{\psi} c_t = (1-\alpha)\frac{y_t}{h_t} \tag{1.30}$$

$$y_t = a_t k_t^{\alpha} h_t^{1-\alpha} \tag{1.31}$$

$$y_t = c_t + i_t \tag{1.32}$$

$$k_{t+1} = i_t + (1-\delta)k_t \tag{1.33}$$

$$1 = \beta E_t \left(\frac{c_t}{c_{t+1}} \left(\alpha \frac{y_{t+1}}{k_{t+1}} + 1 - \delta \right) \right) \tag{1.34}$$

and the transversality condition

$$\lim_{s\to\infty} \beta^s \frac{k_{t+1+s}}{c_{t+s}} = 0$$

The problem with this dynamic system is that it is fundamentally non–linear and therefore the methods we have developed so far are not designed to handle it. The usual way to deal with this type of system is then to take a linear or log–linear approximation of each equation about the deterministic steady state. Therefore, the first step is to find the deterministic steady state.

Deterministic steady state Recall that the steady state value of a variable, x, is the value $x^{\star}$ such that $x_t = x^{\star}$ for all t. Therefore, the steady state of the RBC model is characterized by the set of equations:

$$\Psi h^{\star\psi} c^{\star} = (1-\alpha)\frac{y^{\star}}{h^{\star}} \tag{1.35}$$

$$y^{\star} = \overline{a} k^{\star\alpha} h^{\star 1-\alpha} \tag{1.36}$$

$$y^{\star}{}_t = c^{\star} + i^{\star} \tag{1.37}$$

$$k^{\star} = i^{\star} + (1-\delta)k^{\star} \tag{1.38}$$

$$1 = \beta E_t \left(\left(\alpha \frac{y^{\star}}{k^{\star}} + 1 - \delta \right) \right) \tag{1.39}$$

From equation (1.38), we get

$$i^\star = \delta k^\star \Longleftrightarrow \frac{i^\star}{y^\star} = \delta \frac{k^\star}{y^\star}$$

Then, equation (1.39) implies

$$\frac{y^\star}{k^\star} = \frac{1-\beta(1-\delta)}{\alpha\beta}$$

such that from the previous equation and (1.37)

$$s_i \equiv \frac{i^\star}{y^\star} = \frac{\alpha\beta\delta}{1-\beta(1-\delta)} \Longrightarrow s_c =\equiv \frac{c^\star}{y^\star} = 1 - s_i$$

Then, from (1.35), we obtain

$$h^\star = \left(\frac{1-\alpha}{\Psi s_c}\right)^{\frac{1}{1+\psi}}$$

Finally, it follows from the production function and the definition of $y^\star/k^\star$ that

$$y^\star = \overline{a}\left(\frac{\alpha\beta}{1-\beta(1-\delta)}\right)^{\frac{\alpha}{1-\alpha}} h^\star,\ c^\star = s_c y^\star,\ i^\star = y^\star - c^\star.$$

We are now in position to log–linearize the dynamic system.

Log–linearization: A common practice in the macro literature is to take a log–linear approximation to the equilibrium. Such an approximation is usually taken because it delivers a natural interpretation of the coefficients in front of the variables: these can be interpreted as elasticities. Indeed, let's consider the following onedimensional function $f(x)$ and let's assume that we want to take a log–linear approximation of f around x. This would amount to have, as deviation, a log–deviation rather than a simple deviation, such that we can define

$$\widehat{x} = \log(x) - \log(x^\star)$$

Then, a restatement of the problem is in order, as we are to take an approximation with respect to $\log(x)$:

$$f(x) \simeq f(\exp(\log(x)))$$

which leads to the following first order Taylor expansion

$$f(x) \simeq f(x^\star) + f'(exp(log(x^\star)))exp(log(x^\star))\widehat{x} = f(x^\star) + f'(x^\star)x^\star\widehat{x}$$

Now, remember that by definition of the deterministic steady state, we have $f(x^\star)=0$, such that the latter equation reduces to

$$f(x)\simeq f'(x^\star)x^\star\widehat{x}$$

Applying this technic to the system (1.30)–(1.34), we end up with the system

$$(1+\psi)\widehat{h}_t+\widehat{c}_t-\widehat{y}_t \tag{1.40}$$
$$\widehat{y}_t-(1-\alpha)\widehat{h}_t-\alpha\widehat{h}_t-\widehat{a}_t=0 \tag{1.41}$$
$$\widehat{y}_t-s_c\widehat{c}_t-s_i\widehat{i}_t=0 \tag{1.42}$$
$$\widehat{k}_{t+1}-\delta\widehat{i}_t-(1-\delta)\widehat{k}_t=0 \tag{1.43}$$
$$E_t\widehat{c}_{t+1}-\widehat{c}_t-(1-\beta(1-\delta))(E_t\widehat{y}_{t+1}-E_t\widehat{k}_{t+1}) \tag{1.44}$$
$$\widehat{a}_t-\rho\widehat{a}_{t-1}-\widehat{\varepsilon}_t \tag{1.45}$$

Note that only the last three equations of the system involve dynamics, but they depend on variables that are defined in the first three equations. Either we solve the first three equations in terms of the state and co–state variables, or we adapt a little bit the method. We choose the second solution.
Let us define $Y_t=\{\widehat{k}_{t+1},\widehat{a}_t,E_t\widehat{c}_{t+1}\}$ and $X_t=\{\widehat{y}_t,\widehat{c}_t,\widehat{i}_t,\widehat{h}_t\}$. The system can be rewritten as a set of two equations. The first one gathers static equations

$$\Gamma_{\rm X}X_t=\Gamma_{\rm Y}Y_{t-1}+\Gamma_\varepsilon\varepsilon_t+\Gamma_\eta\eta_t$$

where η_t is the vector of expectation errors, which actually reduces to that attached on $\widehat{c}_t$, and

$$\Gamma_{\rm X}=\begin{pmatrix}1&0&0&\alpha-1\\0&1&0&0\\1&-s_c&-s_i&0\\-1&1&0&1\end{pmatrix}\Gamma_{\rm Y}=\begin{pmatrix}\alpha&\rho&0\\0&0&1\\0&0&0\\0&0&0\end{pmatrix}\Gamma_\varepsilon=\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\Gamma_\eta=\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

The second one gathers the dynamic equations

$$\Upsilon^0_{\rm Y}Y_t+\Upsilon^0_{\rm X}E_tX_{t+1}=\Upsilon^1_{\rm Y}Y_{t-1}+\Upsilon^1_{\rm X}+\Upsilon_\varepsilon\varepsilon_t+\Upsilon_\eta\eta_t$$

with

$$\Upsilon^0_{\rm Y} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1-\beta(1-\delta) & 0 & 1 \end{pmatrix} \quad \Upsilon^0_{\rm X} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -(1-\beta(1-\delta)) & 0 & 0 & 0 \end{pmatrix}$$

$$\Upsilon^1_{\rm Y} = \begin{pmatrix} 1-\delta & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \Upsilon^1_{\rm X} = \begin{pmatrix} 0 & 0 & \delta & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\Gamma_\varepsilon = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \Gamma_\eta = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

From the first equation, we obtain

$$X_t = \Pi_{\rm Y} Y_{t-1} + \Pi_\varepsilon \varepsilon_t + \Pi_\eta \eta_t$$

where $\Pi_j = \Gamma_{\rm X}^{-1}\Gamma_j$, $j = \{{\rm Y}, \varepsilon, \eta\}$. Furthermore, remembering that $E_t\varepsilon_{t+1} = E_t\eta_{t+1} = 0$, we have $E_tX_{t+1} = \Pi_{\rm Y}Y_t$. Hence, plugging this result and the first equation in the second equation we get

$$A_0Y_t = A_1Y_{t+1} + B\varepsilon_t + C\eta_t$$

where $A_0 = \Upsilon^0_{\rm Y} + \Upsilon^0_{\rm X}\Pi_{\rm Y}$, $A_1 = \Upsilon^1_{\rm Y} + \Upsilon^1_{\rm X}\Pi_{\rm Y}$, $B = \Upsilon_\varepsilon + \Upsilon^0_{\rm X}\Pi_\varepsilon$ and $C = \Upsilon_\eta + \Upsilon^0_{\rm X}\Pi_\eta$. We then just use the algorithm as described previously.
Then, we make use of the result in proposition 5, to get η_t. Since it turns out that the model is determinate, the expectation error is a function of the fundamental shock ε_t

$$\eta_t = -V_1 D_{11}^{-1} U_1' Q_2 B \varepsilon_t$$

Plugging this result in the equation governing static equations, we end up with

$$X_t = \Pi_{\rm Y} Y_{t-1} + (\Pi_{\rm E} - \Pi_\eta V_1 D_{11}^{-1} U_1' Q_2 B)\varepsilon_t$$

Figure 1.14 then reports the impulse response function to a 1% technology shock. These IRFs are obtained using the set of parameters reported in table 1.2.

MATLAB CODE: THE RBC MODEL

```
clear all   % Clear memory
%
% Structural parameters
%
alpha    = 0.4;
```

Table 1.2: The Real Business Cycle Model: parameters

α	β	δ	ρ	ψ
0.4	0.988	0.025	0.95	0

```
delta   = 0.025;
rho     = 0.95;
beta    = 0.988;
%
% Deterministic Steady state
%
ysk = (1-beta*(1-delta))/(alpha*beta);
ksy = 1/ysk;
si  = delta/ysk;
sc  = 1-si;
% Define:
%
% Y=[k(t+1) a(t+1) E_tc(t+1)]
%
% X=[y,c,i,h]
%
ny  = 3; % # of variables in vector Y
nx  = 4; % # of variables in vector X
ne  = 1; % # of fundamental shocks
nn  = 1; % # of expectation errors
%
% Initialize the Upsilon matrices
%
UX=zeros(nx,nx);
UY=zeros(nx,ny);
UE=zeros(nx,ne);
UN=zeros(nx,nn);

G0Y=zeros(ny,ny);
G1Y=zeros(ny,ny);
G0X=zeros(ny,nx);
G1X=zeros(ny,nx);
GE=zeros(ny,ne);
GN=zeros(ny,nn);
%
% Production function
%
UX(1,1)=1;
UX(1,4)=alpha-1;
UY(1,1)=alpha;
UY(1,2)=rho;
UE(1)=1;
%
% Consumption c(t)=E(c(t)|t-1)+eta(t)
%
```

```
UX(2,2)=1;
UY(2,3)=1;
UN(2)=1;
%
% Resource constraint
%
UX(3,1)=1;
UX(3,2)=-sc;
UX(3,3)=-si;
%
% Consumption-leisure arbitrage
%
UX(4,1)=-1;
UX(4,2)=1;
UX(4,4)=1;
%
% Accumulation of capital
%
G0Y(1,1)=1;
G1Y(1,1)=1-delta;
G1X(1,3)=delta;
%
% Productivity shock
%
G0Y(2,2)=1;
G1Y(2,2)=rho;
GE(2)=1;
%
% Euler equation
%
G0Y(3,1)=1-beta*(1-delta);
G0Y(3,3)=1;
G0X(3,1)=-(1-beta*(1-delta));
G1X(3,2)=1;
%
% Solution
%
% Step 1: solve the first set of equations
%
PIY = inv(UX)*UY;
PIE = inv(UX)*UE;
PIN = inv(UX)*UN;
%
% Step 2: build the standard System
%
A0  = G0Y+G0X*PIY;
A1  = G1Y+G1X*PIY;
B   = GE+G1X*PIE;
C   = GN+G1X*PIN;
%
% Step 3: Call Sims' routine
%
[MY,ME,ETA,MU_]=sims_solve(A0,A1,B,C);
%
```

```
% Step 4: Recover the impact function
%
PIE=PIE-PIN*ETA;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                                                            %
%               Impulse Response Functions                   %
%                                                            %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
nrep    = 20;            % horizon of responses
YS      = zeros(3,nrep);
XS      = zeros(4,nrep);
Shock   = 1;
YS(:,1) = ME*Shock;
XS(:,1) = PIE;
for t=2:nrep;
    YS(:,t) = MY*YS(:,t-1);
    XS(:,t) = PIY*YS(:,t-1);
end
subplot(221);plot(XS(1,:));title('Output');xlabel('Time')
subplot(222);plot(XS(2,:));title('Consumption');xlabel('Time')
subplot(223);plot(XS(3,:));title('Investment');xlabel('Time')
subplot(224);plot(XS(4,:));title('Hours worked');xlabel('Time')
```

1.6.3 A model with indeterminacy

Let us consider the simplest new keynesian model, with the following IS curve

$$y_t = E_t y_{t+1} - \alpha(i_t - E_t\pi_{t+1}) + g_t$$

where y_t denotes output, π_t is the inflation rate, i_t is the nominal interest rate and g_t is a stochastic shock that follows an AR(1) process of the form

$$g_t = \rho_g g_{t-1} + \varepsilon_t^g$$

the model also includes a Phillips curve that relates positively inflation to the output gap

$$\pi_t = \lambda y_t + \beta E_t\pi_{t+1} + u_t$$

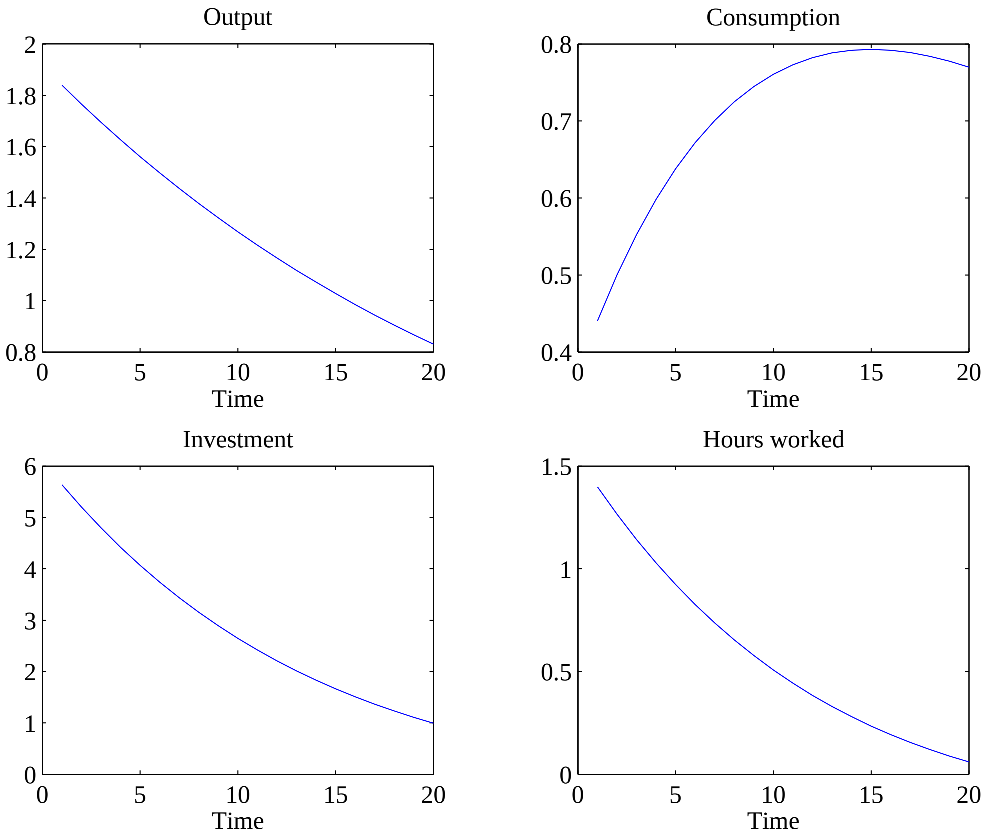
where u_t is a supply shock that obeys

$$u_t = \rho_u u_{t-1} + \varepsilon_t^u$$

For stationarity purposes, we have $|\rho_g| < 1$ and $|\rho_u| < 1$.
The model is closed by a simple Taylor rule of the form

$$i_t = \gamma_\pi \pi_t + \gamma_y y_t$$

Figure 1.14: IRF to a technology shock



Plugging this rule in the first equation, and remembering the definition of expectation errors, the system rewrites

$$\begin{aligned}
y_t =& E_{t-1}y_t + \eta_t^y \\
\pi_t =& E_{t-1}\pi_t + \eta_t^\pi \\
g_t =& \rho_g g_{t-1} + \varepsilon_t^g \\
u_t =& \rho_u u_{t-1} + \varepsilon_t^u \\
(1+\alpha\gamma_y)y_t =& E_t y_{t+1} - \alpha\gamma_\pi \pi_t + \alpha E_t \pi_{t+1} + g_t \\
\pi_t =& \lambda y_t + \beta E_t \pi_{t+1} + u_t
\end{aligned}$$

Defining $Y_t = \{y_t, \pi_t, g_t, u_t, E_t y_{t+1}, E_t \pi_{t+1}\}$ and $\eta_t = \{\eta_t^y, \eta_t^\pi\}$, the system rewrites

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1+\alpha\gamma_y & +\alpha\gamma_\pi & -1 & 0 & -1 & -\alpha \\ -\lambda & 1 & 0 & -1 & 0 & -\beta \end{pmatrix} Y_t = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \rho_g & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho_u & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} Y_{t-1}$$

$$+ \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \varepsilon_t + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \eta_t$$

The set of parameter used in the numerical experiment is reported in table 1.3. As predicted by theory of Taylor rules, a coefficient γ_π below 1 yields indeterminacy.

Table 1.3: New Keynesian model: parameters

α	β	λ	ρ_g	ρ_u	γ_y	γ_π
0.4	0.9	1	0.9	0.9	0.25	1.5/0.5

Matlab Code: A Model with Real Indeterminacy

```
clear all   % Clear memory
%
% Structural parameters
```

```
%
alpha   = 0.4;
gy      = 0.25;
gp      = 0.5;
rho_g   = 0.9;
rho_u   = 0.95;
lambda  = 1;
beta    = 0.9;
% Define:
%
% Y=[y(t),pi(t),g(t),u(t),E_t y(t+1),E_t pi(t+1)]
%
ny  = 6; % # of variables in vector Y
ne  = 2; % # of fundamental shocks
nn  = 2; % # of expectation errors
%
% Initialize the matrices
%
A0  = zeros(ny,ny);
A1  = zeros(ny,ny);
B   = zeros(ny,ne);
C   = zeros(ny,nn);
%
% Output
%
A0(1,1) = 1;
A1(1,5) = 1;
C(1,1)  = 1;
%
% Inflation
%
A0(2,2) = 1;
A1(2,6) = 1;
C(2,2)  = 1;
%
% IS shock
%
A0(3,3) = 1;
A1(3,3) = rho_g;
B(3,1)  = 1;
%
% Supply shock
%
A0(4,4) = 1;
A1(4,4) = rho_u;
B(4,2)  = 1;
%
% IS curve
%
A0(5,1) = 1+alpha*gy;
A0(5,2) = alpha*gp;
A0(5,3) = -1;
A0(5,5) = -1;
A0(5,6) = -alpha;
```

```
%
% Phillips Curve
%
A0(6,1) = -lambda;
A0(6,2) = 1;
A0(6,4) = -1;
A0(6,6) = -beta;
%
% Call Sims' routine
%
[MY,ME,ETA,MU_]=sims_solve(A0,A1,B,C);
```

1.6.4 AK growth model

Up to now, we have considered quadratic objective function in order to get linear expectational difference equations. This may seem to be very restrictive. However, there is a number of situations, where the dynamics generated by the model is characterized by a linear expectational difference equation, despite the objective function is not quadratic. We provide you with such an example in this section.

We consider an endogenous growth model à la Romer [1986] extended to a stochastic environment. The economy consists of a large number of dynastic households and a large number of firms. Firms are producing a homogeneous final product that can be either consumed or invested by means of capital, but contrary to the standard optimal growth model, returns to factors that can be accumulated (namely capital) are exactly constant.

Household decides on consumption, C_t, and capital accumulation (or savings), K_{t+1}, maximizing her lifetime expected utility

$$\max E_t \sum_{s=0}^{\infty} \beta^s \log(C_{t+s})$$

subject to the resource constraint in the economy

$$Y_t = C_t + I_t$$

and the law of motion of capital

$$K_{t+1} = I_t + (1-\delta)K_t \text{ with } \delta \in [0;1]$$

I_t is investment, Y_t denotes output, which is produced using a linear technology of the form $Y_t = A_t K_t$. A_t is a stochastic shock that we leave unspecified for the moment. We may think of it as a shift on the technology, such that it represents a technology shock.

First order conditions: We now present the derivation of the optimal behavior of the consumer. The first order condition associated to the consumption/savings decisions may be obtained forming the following Lagrangean, where Λ_t is the multiplier associated to the resource constraint

$$\mathcal{L}_t = E_t \sum_{s=0}^{\infty} \beta^s \log(C_{t+s}) + \Lambda_t \left(A_t K_t + (1-\delta)K_t - C_t - K_{t+1}\right)$$

Terms involving C_t:

$$\max_{\{C_t\}} E_t \left(\log(C_t) - \Lambda_t C_t\right) = \max_{\{C_t\}} \left(\log(C_t) - \Lambda_t C_t\right)$$

Therefore, the FOC associated to consumption writes

$$\frac{1}{C_t} = \Lambda_t$$

Likewise for the saving decision, terms involving K_{t+1}:

$$\max_{\{K_{t+1}\}} -\Lambda_t K_{t+1} + \beta E_t \left[\Lambda_{t+1} \left(A_{t+1} K_{t+1} + (1-\delta)K_{t+1}\right)\right]$$

such that the FOC is given by

$$\Lambda_t = \beta E_t \left[\Lambda_{t+1} \left(A_{t+1} + 1 - \delta\right)\right]$$

Finally, we impose the so–called transversality condition

$$\lim_{T \longrightarrow \infty} \beta^T E_t \left(\frac{K_{T+1}}{C_T}\right) = 0$$

Solving the dynamic system: Plugging the first order condition on consumption in the Euler equation, we get

$$\frac{1}{C_t} = \beta E_t \left[\frac{1}{C_{t+1}} \left(A_{t+1} + 1 - \delta\right)\right]$$

This system seems to be non–linear, but we can make it linear very easily. Indeed, let us multiply both sides of the Euler equation by K_{t+1}, we get

$$\frac{K_{t+1}}{C_t} = \beta E_t \left[\frac{K_{t+1}}{C_{t+1}} \left(A_{t+1} + 1 - \delta\right)\right]$$

But the resource constraint states that

$$K_{t+1} + C_t = K_t\,(A_t + 1 - \delta) \Longleftrightarrow K_{t+2} + C_{t+1} = K_{t+1}\,(A_{t+1} + 1 - \delta)$$

such that the Euler equation rewrites

$$\frac{K_{t+1}}{C_t} = \beta E_t\left[\frac{K_{t+2} + C_{t+1}}{C_{t+1}}\right] = \beta E_t\left[1 + \frac{K_{t+2}}{C_{t+1}}\right]$$

Let us denote $X_t = K_{t+1}/C_t$, the latter equation rewrites

$$X_t = \beta E_t(1 + X_{t+1})$$

which has the same form as (1.2). As we have already seen, the solution for such an equation can be easily obtained iterating forward. We then get

$$X_t = \beta \lim_{T\to\infty}\sum_{k=0}^{T}\beta^k + \lim_{T\to\infty}\beta^T E_t(X_{T+1})$$

The second term in the right hand side of the latter equation corresponds precisely to the transversality condition. Hence, X_t reduces to

$$X_t = \frac{\beta}{1-\beta} \Longleftrightarrow K_{t+1} = \frac{\beta}{1-\beta}C_t$$

Plugging this relation in the resource constraint, we get

$$K_{t+1} = \beta(A_t + 1 - \delta)K_t$$

and

$$C_t = (1-\beta)(A_t + 1 - \delta)K_t$$

Time series properties Let us consider the solution for capital accumulation. Taking logs, we get

$$\log(K_{t+1}) = \log(K_t) + \log(\beta(A_t + 1 - \delta))$$

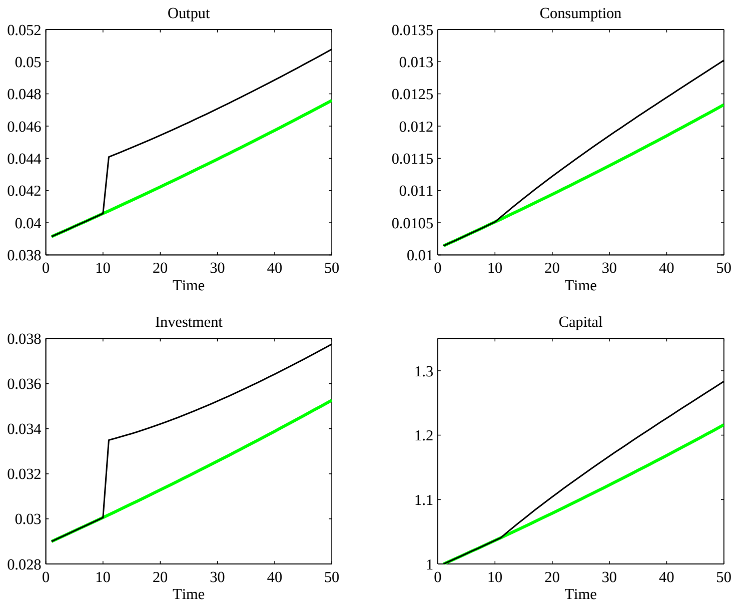
since A_t is an exogenous stochastic process, we immediately see that the process may be rewritten as

$$\log(K_{t+1}) = \log(K_t) + \eta_t$$

where $\eta_t \equiv \log(\beta(A_t + 1 - \delta))$. Such that we see that capital is an non–stationary process (an I(1) process) — more precisely a random walk. Since consumption, output and investment are just a linear function of capital, the non–stationarity of capital translates into the non stationarity of these variables. Nevertheless, as can be seen from the law of motion of consumption, for example, $\log(C_t) - \log(K_t)$ is a stationary process. K_t and C_t are then said to be cointegrated with a cointegrating vector $(1, -1)$.

This has extremely important economic implications, that may be analyzed in the light of the impulse response functions, reported in figure 1.15. In fact, figure 1.15 reports two balanced growth paths for each variable: The first one corresponds to the path without any shock, the second one corresponds to the path that includes a non expected positive shock on technology in period 10. As can be seen, this shock yields a permanent increase in all variables. Therefore, this model can account for the fact that countries may not converge. *Why is that so?* The answer to this question is actually simple and may be

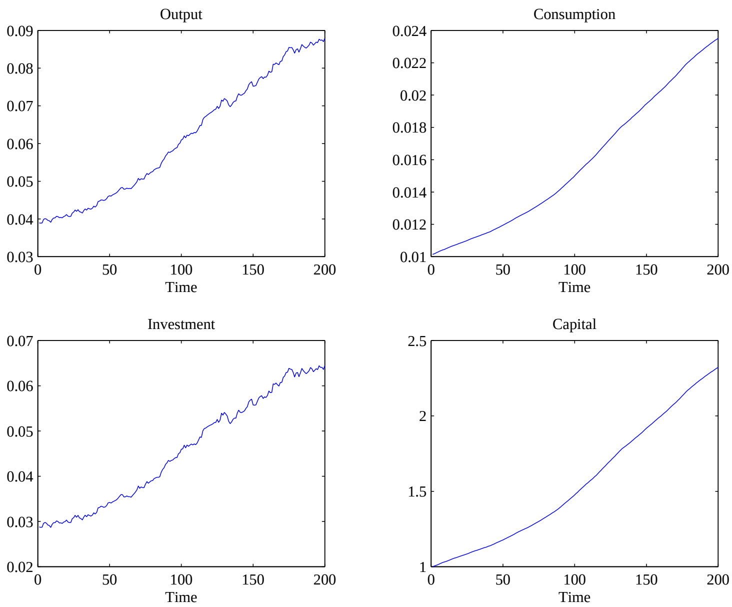
Figure 1.15: Impulse response functions



understood if we go back to the simplest Solow growth model. Assume there is a similar shock in the Solow growth model, output increases on impact and since income increases so does investment yielding higher accumulation. Because the technology displays decreasing returns to capital in the solow growth model, the marginal efficiency of capital decreases reducing incentives to investment so that capital accumulation slows down. The economy then goes back to its steady state. Things are different in this model: Following a shock, income increases. This triggers faster accumulation, but since the marginal productivity of capital is totally determined by the exogenous shock, there is no endogenous force that can drive the economy back to its steady state. Therefore, each additional capital is kept forever.

This implies that following shocks, the economy will enter an ever growing regime. This may be seen from figure 1.16 which reports a simulated path for each variable. These simulated data may be used to generate time moments on

Figure 1.16: Simulated data



the rate of growth of each variable, which estimates are reported in table (1.4) and which distributions are represented in figures 1.17–1.20. It is interesting to note that all variables exhibit — when taken in log–levels — a spurious correlation with output that just reflects the existence of a common trend due to the balanced growth path hypothesis.

Table 1.4: Monte–Carlo Simulations

	ΔY	ΔC	ΔI	ΔK
E	0.40	0.40	0.40	0.40
σ	0.79	0.09	1.06	0.09
Corr(.,ΔY)	1.00	0.30	0.99	-0.08
ρ	-0.01	0.93	-0.02	0.93
	Y	C	I	K
Corr(.,Y)	0.99	0.99	0.99	0.99

Figure 1.17: Rates of growth: distribution of mean

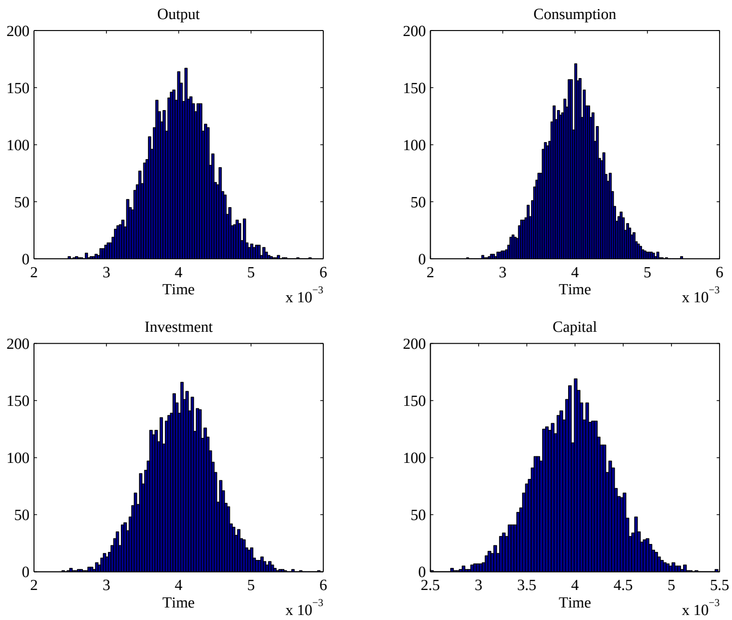


Figure 1.18: Rates of growth: distribution of standard deviation

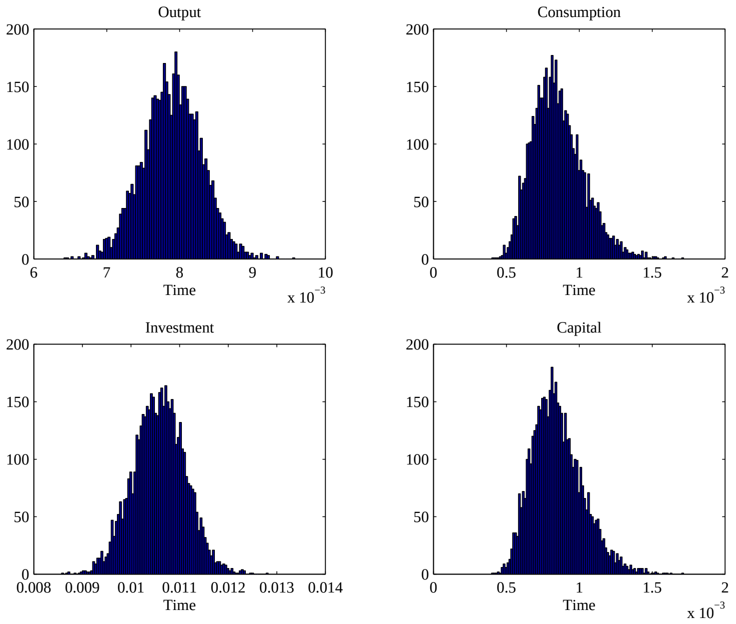


Figure 1.19: Rates of growth: distribution of correlation with ΔY

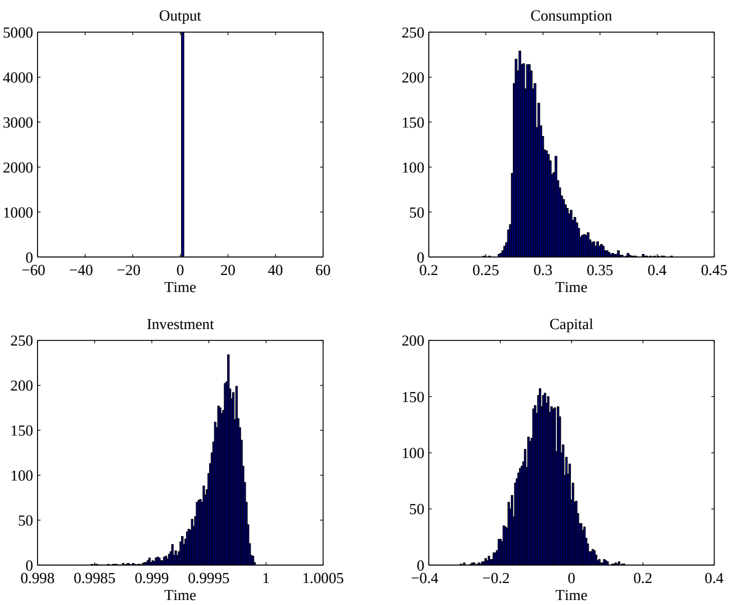
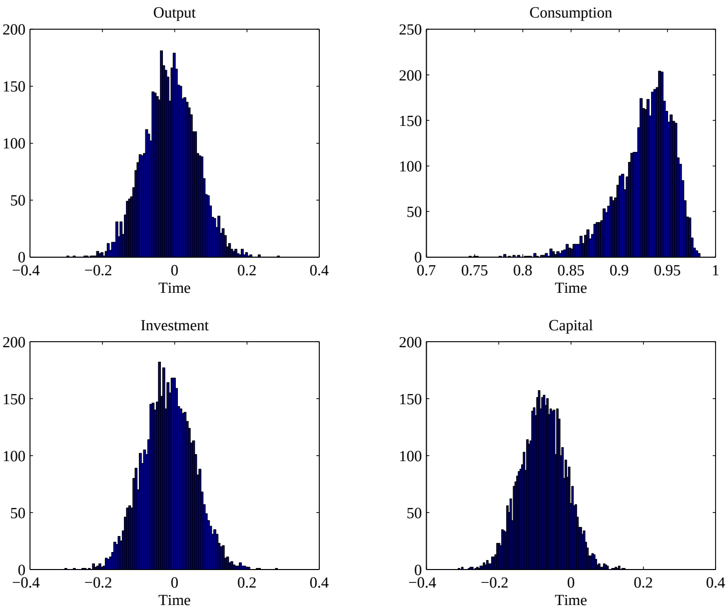


Figure 1.20: Rates of growth: distribution of first order autocorrelation



Matlab Code: AK growth model

```
%
% AK growth model
%
long    = 200;
nsim    = 5000;
nrep    = 50;
%
% Structural parameters
%
gx      = 1.004;
beta    = 0.99;
delta   = 0.025;
rho     = 0.95;
se      = 0.0079;
ab      = (gx-beta*(1-delta))/beta;
K0      = 1;
%
% IRF
%
K1(1)   = K0;
K2(1)   = K0;
a2      = zeros(nrep,1);
K1      = zeros(nrep,1);
K2      = zeros(nrep,1);
K1(1)   = K0;
K2(1)   = K0;
a2(1)   = log(ab);
e       = zeros(nrep,1);
e(11)   = 10*se;
T=[1:nrep];
for i   = 2:nrep;
  a2(i)= rho*a2(i-1)+(1-rho)*log(ab)+e(i);
  K1(i)= beta*(ab+1-delta)*K1(i-1);
  K2(i)= beta*(exp(a2(i-1))+1-delta)*K2(i-1);
end;
C1 = (1-beta)*(ab+1-delta).*K1;
Y1 = ab*K1;
I1 = Y1-C1;
C2 = (1-beta)*(exp(a2)+1-delta).*K2;
Y2 = exp(a2).*K2;
I2 = Y2-C2;
Y=[Y1(:) Y2(:)];
C=[C1(:) C2(:)];
K=[K1(:) K2(:)];
I=[I1(:) I2(:)];
%
% Simulations
%
cx=zeros(nsim,4);
mx=zeros(nsim,4);
sx=zeros(nsim,4);
rx=zeros(nsim,4);
```

```
for s   = 1:nsim;
  disp(s)
  randn('state',s);
  e         = randn(long,1)*se;
  a         = zeros(long,1);
  K         = zeros(long,1);
  a(1)   = log(ab)+e(1);
  K(1)   = K0;
  for   i   = 2:long;
    a(i)= rho*a(i-1)+(1-rho)*log(ab)+e(i);
    K(i)= beta*(exp(a(i-1))+1-delta)*K(i-1);
  end;
  C         = (1-beta)*(exp(a)+1-delta).*K;
  Y         = exp(a).*K;
  I         = Y-C;
  X         = [Y C I K];
  dx        = diff(log(X));
  mx(s,:)   = mean(dx);
  sx(s,:)   = std(dx);
  tmp       = corrcoef(dx);cx(s,:)=tmp(1,:);
  tmp       = corrcoef(dx(2:end,1),dx(1:end-1,1));ry=tmp(1,2);
  tmp       = corrcoef(dx(2:end,2),dx(1:end-1,2));rc=tmp(1,2);
  tmp       = corrcoef(dx(2:end,3),dx(1:end-1,3));ri=tmp(1,2);
  tmp       = corrcoef(dx(2:end,4),dx(1:end-1,4));rk=tmp(1,2);
  rx(s,:)   = [ry rc ri rk];
end;
disp(mean(mx))
disp(mean(sx))
disp(mean(cx))
disp(mean(rx))
```

1.6.5 Announcements

In the last two examples, we will help you to give an answer to this crucial question:

"Why do these two guys annoy us with rational expectations?"

In this example we will show you how different may the impulse response to a shock be different depending on the fact that the shock is announced or not. To illustrate this issue, let us go back to the problem of asset pricing. Let p_t be the price of a stock, d_t be the dividend — which will be taken as exogenous — and r be the rate of return on a riskless asset, assumed to be held constant over time. As we have seen earlier, standard theory of finance states that when agents are risk neutral, the asset pricing equation is given by:

$$\frac{E_t p_{t+1} - p_t}{p_t} + \frac{d_t}{p_t} = r$$

or equivalently

$$p_t = \frac{1}{1+r}E_t p_{t+1} + \frac{1}{1+r}d_t$$

Let us now consider that the dividend policy of the firm is such that from period 0 on, the firm serves a dividend equal to d_0. The price of the asset is therefore given by

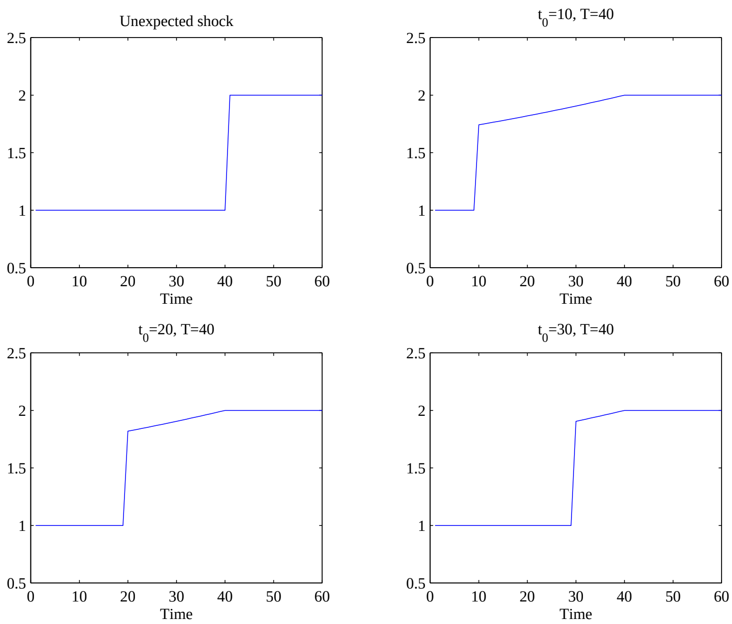
$$p_t = \frac{1}{1+r}\sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^i E_t d_{t+i} = \frac{d_0}{r}$$

If, in period T, the firm unexpectedly decides to serve a dividend of $d_1 > d_0$, the price of the asset will be given by

$$p_t = \frac{d_1}{r} \; \forall t > T$$

In other words, the price of the asset shifts upward to its new level, as shown in the upper–left panel of figure 1.21. Let us now assume that the firm an-

Figure 1.21: Asset pricing behavior



nounces in $t_0 < T$ that it will raise its dividend from d_0 to d_1 in period T. This dramatically changes the behavior of the asset price, as the structure of information is totally modified. Indeed, before the shock is announced by the firm, the level of the asset price establishes at

$$p_t = \frac{d_0}{r}$$

as before. In period t_0 things change as the individuals now know that in $T-t_0$ period the price will be different, this information is now included in the information set they use to formulate expectations. Hence, from period t_0 to T, they take this information into account in their calculation, and the asset price is now given by

$$\begin{aligned}
p_t &= \frac{1}{1+r}\sum_{i=t}^{T-1}\left(\frac{1}{1+r}\right)^{i-t}d_0+\frac{1}{1+r}\sum_{i=T}^{\infty}\left(\frac{1}{1+r}\right)^{i-t}d_1\\
&= \frac{1}{1+r}\sum_{i=t}^{T-1}\left(\frac{1}{1+r}\right)^{i-t}d_0+\frac{1}{1+r}\sum_{i=T}^{\infty}\left(\frac{1}{1+r}\right)^{i-t}(d_1-d_0+d_0)\\
&= \frac{1}{1+r}\sum_{i=t}^{\infty}\left(\frac{1}{1+r}\right)^{i-t}d_0+\frac{1}{1+r}\sum_{i=T}^{\infty}\left(\frac{1}{1+r}\right)^{i-t}(d_1-d_0)
\end{aligned}$$

Denoting $j=i-t$ in the first sum and $\ell=i-T$ in the second, we have

$$\begin{aligned}
p_t &= \frac{1}{1+r}\sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^{j}d_0+\frac{1}{1+r}\sum_{\ell=0}^{\infty}\left(\frac{1}{1+r}\right)^{\ell+T-t}(d_1-d_0)\\
&= \frac{d_0}{r}+\left(\frac{1}{1+r}\right)^{T-t}\left(\frac{d_1-d_0}{r}\right)
\end{aligned}$$

Finally, from T on, the shock has taken place, such that the value of the asset is given by

$$p_t=\frac{d_1}{r}$$

Hence, the dynamics of the asset price is given by

$$p_t=\begin{cases}\frac{d_0}{r} & \text{for } t<t_0\\ \frac{d_0}{r}+\left(\frac{1}{1+r}\right)^{T-t}\left(\frac{d_1-d_0}{r}\right) & \text{for } t_0\leqslant t\leqslant T\\ \frac{d_1}{r} & \text{for } t>T\end{cases}$$

Hence, compared to the earlier situation, there is now a transition phase that takes place as soon as the individuals has learnt the news and exploits this additional piece of information when formulating her expectations. This dynamics is depicted in figure 1.21 for different dates of announcement.

1.6.6 The Lucas critique

As a last example, we now have a look at the so–called Lucas critique. One typical answer to the question raised in the previous section may be found in the so–called Lucas critique (see e.g. Lucas [1976]) , or the *econometric policy evaluation critique*, which asserts that because the *apparently* (for old–fashioned econometricians) structural parameters of a model may change when policy changes, standard econometrics may not be used to study alternative regimes. In order to illustrate this point, let us go back to the simplest example we were dealing with:

$$\begin{aligned} y_t &= aE_t y_{t+1} + bx_t \\ x_t &= \rho x_{t-1} + \varepsilon_t \end{aligned}$$

which solution is given by

$$y_t = \frac{b}{1-a\rho} x_t$$

Now let us assume for a while that y_t denotes output and x_t is money, which is discretionary provided by a central bank. An econometrician that has access to data on output and money would estimate the reduced form of the model

$$y_t = \alpha x_t$$

where $\widehat{\alpha}$ should converge to $b/(1-a\rho)$. Now the central banker would like to evaluate the implications of a new monetary policy from $t = T$ on

$$x_t = \theta x_{t-1} + \varepsilon_t \text{ with } \theta > \rho$$

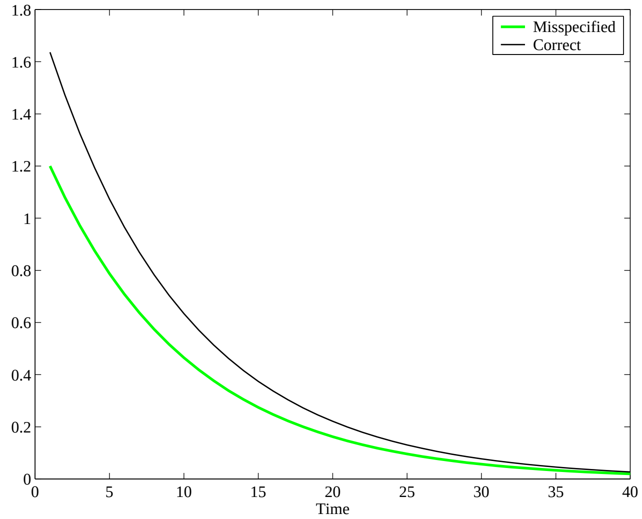
What should be done then? The old–fashioned econometrician would do the following:

1. Take the estimated reduced form: $y_t = \widehat{\alpha} x_t$
2. Simulate paths for the new x_t process
3. Analyse the properties of the time series

Stated like that all seems OK. But such an approach is totally false. Indeed, underlying the rational expectations hypothesis is the fact that the agents know the overall structure of the model, therefore, the agents know that from $t = T$ on the new monetary policy is

$$x_t = \theta x_{t-1} + \varepsilon_t \text{ with } \theta > \rho$$

Figure 1.22: The Lucas Critique



the model needs then to be solved again to yield

$$y_t = \frac{b}{1 - a\theta} x_t$$

Therefore the econometrician should re–estimate the reduced form to get

$$y_t = \widehat{\beta} x_t$$

Keeping the old reduced form implies a systematic bias of

$$\frac{ab(\theta - \rho)}{(1 - a\rho)(1 - a\theta)}$$

To give you an idea of the type of mistake one may do, we report in figure 1.22 the impulse response to a monetary shock in the second monetary rule when the old reduced form (misspecified) and the new one (correct) are used. As it should be clear to you using the wrong rule leads to a systematic bias in policy evaluation since it biases — in this case — the impact effect of the policy. Why is that so? Because the rational expectations hypothesis implies that the expectation function is part of the solution of the model.Keep in mind that solving a RE model amounts to find the expectation function.

Hence, from an econometric point of view, the rational expectations hypothesis has extremely important implications since they condition the way we should think of the model, solve the model and therefore evaluate and test the model. This will be studied in the next chapter.

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