Lecture Notes 6

Perturbation methods

In these lecture notes, we will study the so–called perturbation method, a class of method the linear approximation belongs to. The basic idea of this approach is to include higher order terms in the approximation, and therefore take both curvature and risk into account — therefore accounting for the two first points we raised in lecture notes 2. For further insights on these methods see Judd [1998], Judd and Gaspard [1997], Sims [2000] or Schmitt-Grohe and Uribe [2001].

6.1 The general method

This presentation is mainly based on the paper by Schmitt-Grohe and Uribe [2001], who present the quadratic perturbation method at the order 2.

6.1.1 Model representation

The set of equilibrium conditions of a wide variety of RE models takes the form

$$E_t F(y_{t+1}, y_t, x_{t+1}, x_t) = 0$$

where $F : \mathbb{R}^{n_y} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \longrightarrow \mathbb{R}^n$. The state vector x_t is of size $(n_x \times 1)$ and the co–state vector y_t is of size $(n_y \times 1)$. The total number or variables is therefore given by $n = n_x + n_y$. We assume that the state vector may be

partitioned as $x_t = [x_t^1; x_t^2]$, where x^1 consists of endogenous state variables, whereas x^2 consists of exogenous state variables. In order to simplify, let us assume that x^2 follows the process

$$x_{t+1}^2 = Mx_t^2 + \eta\Sigma\varepsilon_{t+1}$$

where ε_{t+1} is $(n_\varepsilon \times 1)$ and is distributed as a $\mathcal{N}(0, I)$. All eigenvalues of M are assumed to have modulus less than one.

The solution to this model is of the form:

$$y_t = g(x_t, \eta) \tag{6.1}$$

$$x_{t+1} = h(x_t, \eta) + \eta\Omega\varepsilon_{t+1} \text{ with } \Omega = \begin{pmatrix} 0 \\ \Sigma \end{pmatrix} \tag{6.2}$$

where g maps $\mathbb{R}^{n_x} \times \mathbb{R}_+$ into $\mathbb{R}^{n_y}$ and h maps $\mathbb{R}^{n_x} \times \mathbb{R}_+$ into $\mathbb{R}^{n_x}$. These are the true solution of the model, such that the model may be rewritten as

$$E_t F\left(g(x_{t+1}, \eta), g(x_t, \eta), h(x_t, \eta) + \eta\Omega\varepsilon_{t+1}, x_t\right) = 0$$
$$E_t F\left(g\left(h(x_t, \eta) + \eta\Omega\varepsilon_{t+1}, \eta\right), g(x_t, \eta), h(x_t, \eta) + \eta\Omega\varepsilon_{t+1}, x_t\right) = 0$$

In order to save on notations, we will now get rid off time subscripts, and a prime will stand for $t + 1$, such that the model rewrites

$$E_t F\left(g\left(h(x, \eta) + \eta\Omega\varepsilon', \eta\right), g(x, \eta), h(x, \eta) + \eta\Omega\varepsilon', x\right) = 0$$

or

$$E_t \mathcal{F}(x, \eta) = 0$$

6.1.2 The Method

Since we cannot compute exactly neither $g(.)$ nor $h(.)$, we shall compute an approximation. But, since we want to take risk and curvature into account, we will take a higher order approximation. Now assume we want to take a higher order approximation for $g(.)$ and $h(.)$. To keep things simple — and you will

see that they are already quite cumbersome — we will restrict ourselves to the second–order approximation, that is

$$\begin{aligned} E_t\mathcal{F}^i(x,\eta) &\simeq E_t\Big(\mathcal{F}^i(x^\star,0)+\nabla_x\mathcal{F}^i(x^\star,0)(x-x^\star)+\nabla_\eta\mathcal{F}^i(x^\star,0)\eta \\ &\quad +\frac{1}{2}(x-x^\star,\eta)\mathcal{H}^i_{\mathcal{F}}\begin{pmatrix} x-x^\star \\ \eta \end{pmatrix}\Big)=0 \end{aligned}$$

for all $i = 1\ldots,n$. Note that we will take the approximation for accurate, such that this is the approximation that will be set to 0. Therefore, for the model to be solved, we need all derivatives, at any order to be 0.

First order approximation: Taking a first order approximation exactly corresponds to what we did at the very beginning of the chapter. Namely, we use the approximation

$$E_t\Big(\mathcal{F}(x^\star,0)+\nabla_x\mathcal{F}(x^\star,0)(x-x^\star)+\nabla_\eta\mathcal{F}(x^\star,0)\eta\Big)=0$$

and we want to set

$$\begin{aligned} \mathcal{F}^i(x^\star,0)&=0 \\ \nabla_x\mathcal{F}^i(x^\star,0)&=0 \\ \nabla_\eta\mathcal{F}^i(x^\star,0)&=0 \end{aligned}$$

for all $i = 1\ldots,n$. The object we are looking for are however somewhat different, since we look for something like

$$\begin{aligned} g(x,\eta) &= g(x^\star,0)+g_x(x^\star,0)(x-x^\star)+g_\eta(x^\star,0)\eta \\ h(x,\eta) &= h(x^\star,0)+h_x(x^\star,0)(x-x^\star)+h_\eta(x^\star,0)\eta \end{aligned}$$

our problem is then to determine $g(x^\star,0)$, $g_x(x^\star,0)$, $g_\eta(x^\star,0)$, $h(x^\star,0)$, $h_x(x^\star,0)$ and $h_\eta(x^\star,0)$. But we already know at least two quantities since by definition

$$\begin{aligned} y^\star &= g(x^\star,0) \\ x^\star &= h(x^\star,0) \end{aligned}$$

Taking the second condition seriously we have to solve

$$[\nabla_x \mathcal{F}(x^\star, 0)]^i_j = [F_{y'}]^i_\alpha [g_x]^\alpha_\beta [h_x]^\beta_j + [F_y]^i_\alpha [g_x]^\alpha_j + [F_{x'}]^i_\beta [h_x]^\beta_j + [F_x]^i_j = 0 \quad (6.3)$$

with $i = 1, \ldots, n$, $\alpha = 1, \ldots, n_y$, and $\beta, j = 1, \ldots, n_x$. Implicit in this formulation is the fact that we are using tensor notation. Writing the Taylor expansion of a multidimensional system is very demanding on notation. We therefore adopt the following representation for partial derivatives, loosely adapted from the tensor notation advocated by Judd [1998]:

$$[F_x]^i_j = \frac{\partial f_i}{\partial x_j}$$

The same index in both superscript and subscript positions indicates a summation. For example,

$$[F_x]^i_\alpha [h_x]^\alpha_j = \sum_\alpha \frac{\partial f_i}{\partial x_\alpha} \frac{\partial h_\alpha}{\partial x_j}$$

Equation (6.3) defines a $(n_y + n_x) \times n_x$ system of equations, sufficient to determine $n_y \times n_x$ partial derivatives of y with respect to x and $n_x \times n_x$ partial derivatives of x' with respect to x. This system of equations is a matrix polynomial equation and is solved under the requirement that the system returns asymptotically to equilibrium in the absence of other future shocks (transversality condition). In other words we are back to a linear RE model of the kind we have solved in the first series of the lectures, such that we may use one of the method we already saw to get $h_x(.)$ and $g_x(.)$.

Similarly, $g_\sigma(.)$ and $h_\sigma(.)$ can be obtained imposing $\nabla_\eta F(x^\star, 0) = 0$. This amounts to solve the system

$$\begin{aligned}[\nabla_\eta \mathcal{F}(x^\star, 0)]^i &= E_t\Big([F_{y'}]^i_\alpha [g_x]^\alpha_\beta [h_\eta]^\beta + [F_{y'}]^i_\alpha [g_x]^\alpha_\beta [\Omega]^\beta_\varphi [\varepsilon']^\varphi + [F_{y'}]^i_\alpha [g_\eta]_\alpha \\ &\quad [F_y]^i_\alpha [g_\eta]_\alpha + [F_{x'}]^i_\beta [h_\eta]^\beta + [F_{x'}]^i_\beta [\Omega]^\beta_\varphi [\varepsilon']^\varphi \Big) = 0\end{aligned}$$

Note that this equation involves expectations of ε_{t+1} which are identically equal to 0, such that the system reduces to

$$[F_{y'}]^i_\alpha [g_x]^\alpha_\beta [h_\eta]^\beta + [F_{y'}]^i_\alpha [g_\eta]^\alpha + [F_y]^i_\alpha [g_\eta]^\alpha + [F_{x'}]^i_\beta [h_\eta]^\beta = 0 \qquad (6.4)$$

with $i = 1, \ldots, n$, $\alpha = 1, \ldots, n_y$, and $\beta, j = 1, \ldots, n_x$. But as may be noticed, this equation is homogenous in g_η and h_η, such that if a unique solution exists, it has to be the case that

$$g_\eta(x^\star, 0) = 0 \text{ and } h_\eta(x^\star, 0) = 0$$

in other words risk does not matter, and we are back to the certainty equivalence property. Now we will expand the solution up to the second order.

Second order approximation: We now want to obtain an approximation for each $g^i(.)$, $i = 1, \ldots, n_y$ and $h^j(.)$, $j = 1, \ldots, n_x$ of the form

$$\begin{aligned} g^i(x, \eta) &= g^i(x^\star, 0) + g^i_x(x^\star, 0)(x - x^\star) + g^i_\eta(x^\star, 0)\eta + \frac{1}{2}(x - x^\star, \eta)\mathcal{H}^i_g \begin{pmatrix} x - x^\star \\ \eta \end{pmatrix} \\ h^j(x, \eta) &= h^j(x^\star, 0) + h^j_x(x^\star, 0)(x - x^\star) + h^j_\eta(x^\star, 0)\eta + \frac{1}{2}(x - x^\star, \eta)\mathcal{H}^j_h \begin{pmatrix} x - x^\star \\ \eta \end{pmatrix} \end{aligned}$$

We have already solved the first order problem, we now need to reveal information on

$$\mathcal{H}^i_g = \begin{pmatrix} g^i_{xx}(x^\star, 0) & \vdots & g^i_{x\eta}(x^\star, 0) \\ \cdots & & \cdots \\ g^i_{\eta x}(x^\star, 0) & \vdots & g^i_{\eta\eta}(x^\star, 0) \end{pmatrix} \text{ and } \mathcal{H}^j_h = \begin{pmatrix} h^j_{xx}(x^\star, 0) & \vdots & h^j_{x\eta}(x^\star, 0) \\ \cdots & & \cdots \\ h^j_{\eta x}(x^\star, 0) & \vdots & h^j_{\eta\eta}(x^\star, 0) \end{pmatrix}$$

To do so, we only need to impose

$$\begin{aligned} E_t F_{xx}(x^\star, 0) &= 0 \\ E_t F_{x\eta}(x^\star, 0) &= 0 \\ E_t F_{\eta x}(x^\star, 0) &= 0 \\ E_t F_{\eta\eta}(x^\star, 0) &= 0 \end{aligned}$$

Let us start with $g_{xx}(x^\star,0)$ and $h_{xx}(x^\star,0)$, these matrices may be identified, imposing

$$E_t F_{xx}(x^\star,0)=0$$

But before going to this point, we need to establish further notations. Like previously, we will rely on the tensor notations, and will adopt the following conventions:

$$\begin{aligned}[F_{xx}]^i_{jk} &= \frac{\partial^2 F_i}{\partial x_j \partial x_k}\\ [F_{xy}]^i_{jk} &= \frac{\partial^2 F_i}{\partial x_j \partial y_k}.\end{aligned}$$

The same index in both superscript and subscript positions indicates a summation. For example,

$$[F_{xx}]^i_{\alpha\beta}\,[h_x]^\alpha_j\,[h_x]^\beta_k=\sum_\alpha\sum_\beta \frac{\partial^2 F_i}{\partial x_\alpha \partial x_\beta}\frac{\partial h_\alpha}{\partial x_j}\frac{\partial h_\beta}{\partial x_k}.$$

Let us work it out now:

$$\begin{aligned}[F_{xx}(x^\star,0)]^i_{jk} &= \Big([F_{y'y'}]^i_{\alpha\gamma}[g_x]^\gamma_\delta[h_x]^\delta_k+[F_{y'y}]^i_{\alpha\gamma}[g_x]^\gamma_k+[F_{y'x'}]^i_{\alpha\delta}[h_x]^\delta_k+[F_{y'x}]^i_{\alpha k}\Big)[g_x]^\alpha_\beta[h_x]^\beta_j\\ &\quad+[F_{y'}]^i_\alpha\left([g_{xx}]^\alpha_{\beta\delta}[h_x]^\delta_k[h_x]^\beta_j+[g_x]^\alpha_\beta[h_{xx}]^\beta_{jk}\right)\\ &\quad+\Big([F_{yy'}]^i_{\alpha\gamma}[g_x]^\gamma_\delta[h_x]^\delta_k+[F_{yy}]^i_{\alpha\gamma}[g_x]^\gamma_k+[F_{yx'}]^i_{\alpha\delta}[h_x]^\delta_k+[F_{yx}]^i_{\alpha k}\Big)[g_x]^\alpha_j\\ &\quad+\Big([F_{x'y'}]^i_{\beta\gamma}[g_x]^\gamma_\delta[h_x]^\delta_k+[F_{x'y}]^i_{\beta\gamma}[g_x]^\gamma_k+[F_{x'x'}]^i_{\beta\delta}[h_x]^\delta_k+[F_{x'x}]^i_{\beta k}\Big)[h_x]^\beta_j\\ &\quad+[F_{xy'}]^i_{j\gamma}[g_x]^\gamma_\delta[h_x]^\delta_k+[F_{xy}]^i_{j\gamma}[g_x]^\gamma_k+[F_{xx'}]^i_{j\delta}[h_x]^\delta_k+[F_{xx}]^i_{jk}\\ &\quad+[F_y]^i_\alpha[g_{xx}]^\alpha_{jk}+[F_{x'}]^i_\beta[h_{xx}]^\beta_{jk}\\ &= 0\end{aligned}$$

for $i=1,\ldots,n$, $\alpha,\gamma=1,\ldots,n_y$ and $j,k,\beta,\delta=1,\ldots,n_x$. Although it looks quite complicated, this system is just a big linear system that we should solve for $g_{xx}(x^\star,0)$ and $h_{xx}(x^\star,0)$, as all the first order derivatives are perfectly known.

Likewise for $g_{\eta\eta}(x^\star, 0)$ and $h_{\eta\eta}(x^\star, 0)$, we just need to impose

$$E_t F_{\eta\eta}(x^\star, 0) = 0$$

which amounts to

$$\begin{aligned} E_t F_{\eta\eta}(x^\star, 0) &= [F_{y'y'}]^i_{\alpha\gamma}[g_x]^\gamma_\delta[\Omega]^\delta_\xi[g_x]^\alpha_\beta[\Omega]^\beta_\varphi[I]^\varphi_\xi \\ &\quad +[F_{y'}]^i_\alpha\left([g_x]^\alpha_\beta[h_{\eta\eta}]^\beta + [g_{xx}]^\alpha_{\beta\delta}[\Omega]^\delta_\xi[\Omega]^\beta_\varphi[I]^\varphi_\xi + [g_{\eta\eta}]^\alpha\right) \\ &\quad +[F_{y'x'}]^i_{\alpha\delta}[\Omega]^\delta_\xi[g_x]^\alpha_\beta[\Omega]^\beta_\varphi[I]^\varphi_\xi + [F_y]^i_\alpha[g_{\eta\eta}]^\alpha + [F_{x'}]^i_\beta[h_{\eta\eta}]^\beta \\ &\quad +[F_{x'y'}]^i_{\beta\gamma}[g_x]^\gamma_\delta[\Omega]^\delta_\xi[\Omega]^\beta_\varphi[I]^\varphi_\xi + [F_{x'x'}]^i_{\beta\delta}[\Omega]^\delta_\xi[\Omega]^\beta_\varphi[I]^\varphi_\xi \\ &= 0 \end{aligned}$$

for $i = 1, \ldots, n$, $\alpha, \gamma = 1, \ldots, n_y$, $\beta, \delta = 1, \ldots, n_x$ and $\varphi, \varepsilon = 1, \ldots, n_\varepsilon$. Like the previous system, although it looks quite complicated, this is just a linear system that has to be solved for $g_{\eta\eta}(x^\star, 0)$ and $h_{\eta\eta}(x^\star, 0)$, as all the first order derivatives are perfectly known.

Finally, we can easily show that $g_{x\eta}(x^\star, 0)$ and $h_{x\eta}(x^\star, 0)$ are both equal to zero, indeed, we have

$$\begin{aligned} E_t F_{\eta x}(x^\star, 0) &= [F_{y'}]^i_\alpha\left([g_x]^\alpha_\beta[h_{\eta x}]^\beta_j + [g_{\eta x}]^\alpha_\gamma[h_x]^\gamma_j\right) + [F_y]^i_\alpha[g_{\eta x}]^\alpha_j + [F_{x'}]^i_\beta[h_{\eta x}]^\beta_j \\ &= 0 \end{aligned}$$

Since this system is homogenous in the unknowns $g_{x\eta}(x^\star, 0)$ and $h_{x\eta}(x^\star, 0)$, if one solution exists, it is zero. Hence

$$g_{x\eta}(x^\star, 0) = 0 \text{ and } h_{x\eta}(x^\star, 0) = 0$$

Obviously, this solution cannot be obtained analytically, and a numerical package like Gauss or Matlab is needed to implement this method. Nevertheless, in order to give you an idea of how we can implement it, we will apply it to our asset–pricing model developed in lecture notes 2 and to the optimal growth model.

6.2 Implementing the method

6.2.1 The asset pricing model

We now discuss the implementation of the perturbation method to the simple asset pricing model of Burnside [1998], and describe how this method can take advantage of the information carried by the higher moments of the distribution of the shocks. As a first step, let us briefly sketch the model again.

The model economy: We consider a frictionless pure exchange economy *à la* Mehra and Prescott [1985] and Rietz [1988] with a single household and a unique perishable consumption good produced by a single "tree". The household can hold equity shares to transfer wealth from one period to another. The problem of a single agent is then to choose consumption and equity holdings to maximize her expected discounted stream of utility, given by

$$E_t \sum_{\tau=0}^{\infty} \beta^\tau \frac{c_{t+\tau}^\theta}{\theta} \text{ with } \theta \in (-\infty, 0) \cup (0, 1] \tag{6.5}$$

subject to the budget constraint

$$p_t e_{t+1} + c_t = (p_t + d_t) e_t \tag{6.6}$$

where $\beta \in (0, 1)$ is the agent's subjective discount factor, c_t is household's consumption of a single perishable good at date t, p_t denotes the price of the equity in period t and e_t is the household's equity holdings in period t. Finally, d_t is the tree's dividend in period t. Dividends are assumed to grow at rate x_t such that :

$$d_t = \exp(x_t) d_{t-1} \tag{6.7}$$

where x_t, the rate of growth of dividends, is assumed to be a Gaussian stationary AR(1) process

$$x_t = (1 - \rho)\overline{x} + \rho x_{t-1} + \varepsilon_t \tag{6.8}$$

where ε is i.i.d. $\mathcal{N}(0, \sigma^2)$ with $|\rho| < 1$. Market clearing requires that $e_t = 1$ so that $c_t = d_t$ in equilibrium. Like in Burnside [1998], let y_t denote the price–dividend ratio, $y_t = p_t/d_t$. Then, condition for the household's problem can be shown to rewrite as

$$y_t = \beta E_t \left[\exp(\theta x_{t+1})(1 + y_{t+1})\right] \tag{6.9}$$

Burnside [1998] shows that the above equation admits an exact solution of the form[1]

$$y_t = \sum_{i=1}^{\infty} \beta^i \exp\left[a_i + b_i(x_t - \overline{x})\right] \tag{6.10}$$

where

$$a_i = \theta \overline{x} i + \frac{\theta^2 \sigma^2}{2(1-\rho)^2}\left[i - \frac{2\rho(1-\rho^i)}{1-\rho} + \frac{\rho^2(1-\rho^{2i})}{1-\rho^2}\right]$$

and

$$b_i = \frac{\theta\rho(1-\rho^i)}{1-\rho}$$

As can be seen from the definition of a_i, σ — the volatility of the shock, directly enters the decision rule, therefore Burnside's [1998] model does not make the certainty equivalent hypothesis: risk matters for asset holdings decisions.

Finding a second–order approximation to the model: First of all let us rewrite the two key equations that define the model:

$$\begin{aligned} y_t &= \beta E_t \left[\exp(\theta x_{t+1})(1 + y_{t+1})\right] \\ x_t &= (1-\rho)\overline{x} + \rho x_{t-1} + \sigma \varepsilon_t \end{aligned}$$

and let us reexpress the model as

$$x_{t+1} = h(x_t, \sigma) \tag{6.11}$$

$$y_t = E_t[f(y_{t+1}, x_{t+1})] \tag{6.12}$$

[1] See appendix **??** for a detailed exposition of the solution.

where $f(y,x) = \beta \exp(\theta x)(1+y)$ and $h(x,\varepsilon) = \rho x + (1-\rho)\overline{x} + \sigma\varepsilon$. In this setting, the deterministic steady state is given by

$$\left\{ \begin{array}{lcl} x^\star = \rho x^\star + (1-\rho)\overline{x} & \Longleftrightarrow & x^\star = \overline{x} \\ y^\star = f(y^\star, x^\star) & \Longleftrightarrow & y^\star = \beta \exp(\theta\overline{x})/(1-\beta\exp(\theta\overline{x})) \end{array} \right. \tag{6.13}$$

The solution to (6.12) is a function $g(.)$ such that $y_t = g(x_t, \sigma)$. Then (6.12) can be rewritten as

$$g(x_t,\sigma) = E_t[f(g(h(x_t,\sigma)), h(x_t,\sigma))] = E_t[F(x_t,\sigma)] \tag{6.14}$$

The n–th order approximation is then determined by

$$\begin{aligned} \sum_{k=0}^{n} \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} g_{k-j,j}\ \widehat{x}_t^{k-j} \sigma^j &= E_t \left[\sum_{k=0}^{n} \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} F_{k-j,j}\ \widehat{x}_t^{k-j} \sigma^j \right] \\ &= \left[\sum_{k=0}^{n} \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} E_t F_{k-j,j}\ \widehat{x}_t^{k-j} \sigma^j \right] \end{aligned} \tag{6.15}$$

where $g_{k-j,j}$ denotes $\left.\frac{\partial g(x_t,\sigma)}{\partial x_t^{k-j} \partial \sigma^j}\right|_{\substack{x_t = x^\star \\ \sigma = 0}}$ and $F_{k-j,j}$ denotes $\left.\frac{\partial^k F(x_t,\sigma)}{\partial x_t^{k-j} \partial \sigma^j}\right|_{\substack{x_t = x \\ \sigma = 0}}$ and $\mu_j = E_t \widehat{\varepsilon}_{t+1}^j$.

Identifying terms by terms, and after some tedious accounting, we obtain a system of n equations:

$$\frac{1}{k!} g_k = \sum_{j=0}^{n-k} \frac{1}{(j+k)!} \binom{k}{j} F_{k,j}\ \mu_j \quad \text{for } k = 0 \ldots n \tag{6.16}$$

It shall be clear that each $F_{k,j}$ is a function of g_k, $k = 0 \ldots n$, as in our case

$$F_{k,j} = \beta \exp(\theta x) \rho^k \left[\theta^{j+k} + \sum_{\ell=0}^{j+k} \binom{j+k}{\ell} \theta^{j+k-\ell} g_\ell \right] \tag{6.17}$$

where $x_{t+1} = h(x_t, \varepsilon_{t+1})$. Therefore, (6.16) together with (6.17) defines a

linear system that should be solved for g_k, $k = 0, \ldots, n$:

$$\left\{\begin{array}{rcl} g_0 & = & \beta \exp(\theta x) \sum_{j=0}^{n} \frac{1}{j!} \left[\theta^j + \sum_{\ell=0}^{j} \binom{j}{\ell} \theta^{j-\ell} g_\ell\right] \mu_j \\ & \vdots & \\ \frac{1}{k!} g_k & = & \sum_{j=0}^{n-k} \frac{1}{(j+k)!} \binom{k}{j} \beta \exp(\theta x) \rho^k \left[\theta^{j+k} + \sum_{\ell=0}^{j+k} \binom{j+k}{\ell} \theta^{j+k-\ell} g_\ell\right] \mu_j \\ & \vdots & \\ g_n & = & \beta \exp(\theta x) \rho^n \left[\theta^n + \sum_{\ell=0}^{n} \binom{n}{\ell} \theta^{n-\ell} g_\ell\right] \end{array}\right. \tag{6.18}$$

It worth noting that the solution of this system does depend on higher moments of the distribution of the exogenous shocks, μ_j. Therefore, the properties of the decision rule (slope and curvature) will depend on these moments. But more importantly, the level of the rule will differ from the certainty equivalent solution, such that the steady state level, $y^\star$, is not given anymore by (6.13) but will be affected by the higher moments of the distribution of the shocks (essentially the volatility for our model as the shocks are assumed to be normally distributed). Also noteworthy is that this approximation is particularly simple in our showcase economy, as the solution only depends on the exogenous variable x_t.

6.2.2 What do we gain?

This section checks the accuracy and evaluates the potential gains of the method presented above. As the model possesses a closed–form solution, we can directly check the accuracy of each solution we propose against this "true" solution. We first present the evaluation criteria that will be used to check the accuracy. We then conduct the evaluation of the approximation methods under study.

Criteria

Several criteria are considered to tackle the question of the accuracy of the different approximation methods under study. As the model admits a closed–form solution, the accuracy of the approximation method can be directly

checked against the "true" decision rule. This is undertaken relying on the two following criteria

$$E_1 = 100 \times \frac{1}{N} \sum_{t=1}^{N} \left| \frac{y_t - \widetilde{y}_t}{y_t} \right|$$

and

$$E_\infty = 100 \times \max \left\{ \left| \frac{y_t - \widetilde{y}_t}{y_t} \right| \right\}$$

where y_t denotes the true solution to price–dividend ratio and $\widetilde{y}_t$ is the approximation of the true solution by the method under study. E_1 represents the average relative error an agent makes using the approximation rather than the true solution, while E_∞ is the maximal relative error made using the approximation rather than the true solution. These criteria are evaluated over the interval $x_t \in [\overline{x} - \Delta\sigma_x, \overline{x} + \Delta\sigma_x]$ where Δ is selected such that we explore 99.99% of the distribution of x.

However, one may be more concerned — especially in finance — with the ability of the approximation method to account for the distribution of the PDR, and therefore the moments of the distribution. We first compute the mean of y_t for different calibration and different approximation methods. Further, we explore the stochastic properties of the innovation of y_t: $\zeta_t = y_t - E_{t-1}(y_t)$, in order to assess the ability of each approximation method to account for the internal stochastic properties of the model. We thus report standard deviation, skewness and kurtosis of ζ_t, which provides information on the ability of the model to capture the heteroskedasticity of the innovation and more importantly the potential non–linearities the model can generate.[2]

Approximation errors

Table 6.1 reports E_1 and E_∞ for the different cases and different approximation methods under study. We also consider different cases. Our benchmark

[2]The cdf of ζ_t is computed using 20000 replications of Monte–Carlo simulations of 1000 observations for ζ_t.

experiment amounts to considering the Mehra and Prescott's [1985] parameterization of the asset pricing model. We therefore set the mean of the rate of growth of dividend to $\overline{x} = 0.0179$, its persistence to $\rho = -0.139$ and the volatility of the innovations to $\sigma = 0.0348$. These values are consistent with the properties of consumption growth in annual data from 1889 to 1979. θ was set to -1.5, the value widely used in the literature, and β to 0.95, which is standard for annual frequency. We then investigate the implications of changes in these parameters in terms of accuracy. In particular, we study the implications of larger and lower impatience, higher volatility, larger curvature of the utility function and more persistence in the rate of growth of dividends.

At a first glance at table 6.1, it appears that linear approximation can only accommodate situations where the economy does not experiment high volatility or large persistence of the growth of dividends, or where the utility of individuals does not exhibit much curvature. This is for instance the case in the Mehra and Prescott's [1985] parameterization (benchmark) case as both the average and maximal approximation error lie around 1.5%. But, as is nowadays well–known, increases along one of the aforementioned dimension yields lower accuracy of the linear approximation. For instance, increasing the volatility of the innovations of the rate of growth of dividends to σ=0.1 yields approximation errors of almost 12% both in average and at the maximum, thus indicating that the approximation performs particularly badly in this case. This is even worse when the persistence of the exogenous process increases, as ρ=0.9 yields an average approximation error of about 60% and a maximal approximation of about 225%. This is also true for increases in the curvature of the utility function (see row 4 and 5 of table 6.1).

If we now apply higher order Taylor expansions as proposed in section **??**, the gains in accuracy are particularly significant. In most of the cases under study, the approximation error is reduced to less than 0.10%. For instance, in the benchmark case, the linear approximation led to a 1.5% error, the

Table 6.1: Accuracy check

	Linear		Quadratic (CE)		Quadratic	
	E_1	E_∞	E_1	E_∞	E_1	E_∞
Benchmark	1.4414	1.4774	1.4239	1.4241	0.0269	0.0642
β=0.5	0.2537	0.2944	0.2338	0.2343	0.0041	0.0087
β=0.99	2.9414	2.9765	2.9243	2.9244	0.0833	0.1737
θ=-10	23.7719	25.3774	23.1348	23.1764	4.5777	8.3880
θ=0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
θ=0.5	0.2865	0.2904	0.2845	0.2845	0.0016	0.0038
σ=0.001	0.0012	0.0012	0.0012	0.0012	0.0000	0.0000
σ=0.1	11.8200	12.1078	11.6901	11.6933	1.2835	2.2265
ρ=0	1.8469	1.8469	1.8469	1.8469	0.0329	0.0329
ρ=0.5	5.9148	8.2770	4.9136	5.2081	0.7100	1.5640
ρ=0.9	57.5112	226.2032	31.8128	146.6219	36.8337	193.1591

Note: The series defining the true solution was truncated after 800 terms, as no significant improvement was found adding additional terms at the machine accuracy. When exploring variations in ρ, the overall volatility of the rate of growth of dividends was maintained to its benchmark level.

quadratic approximation leads to a less than 0.1% error. This error is even lower for low volatility economies, and is essentially zero in the case of σ=0.001. Even for higher volatility, the gains from applying a quadratic approximation can be substantial, as the case $\sigma = 0.1$ shows. In effect, the average error drops from 12% in the linear case to 1.3% in the quadratic case. The gain is also important for low degrees of intertemporal substitution as the error is less than 5% for θ=-10 compared to the 25% previously obtained.

The gains seem to be impressive, but something still remains to be understood: *where does this gain come from?*

Figure 6.1 sheds light on these results. We consider a rather extreme situation where $\theta = -5$, $\rho = -0.5$ and the volatility of the shock is preserved. In this graph, Linear corresponds to the linear approximation of the true decision rule, quadratic (CE) is the quadratic approximation but ignores the correction introduced by the volatility — and therefore can be qualified as a certainty equivalent quadratic approximation — and quadratic takes care of the correction.

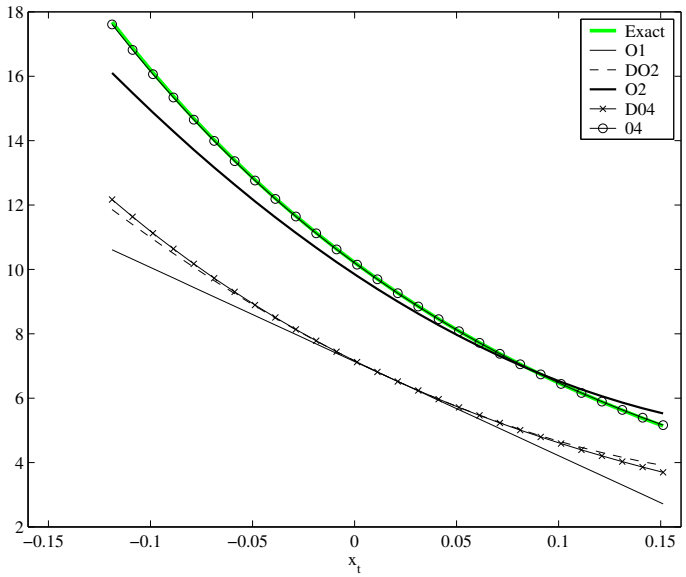
As can be seen from figure 1, the major gains from moving from a linear to a "non–stochastic" higher order Taylor expansion (CE) is found in the ability of the latter to take care of the curvature of the decision rule. But this is not a sufficient improvement as far as accuracy of the solution is concerned. This is confirmed by table 6.2,which reports the approximation errors in each case. It clearly indicates that the gain from increasing the order of approximation,

Table 6.2: Accuracy check (θ=-5, ρ=-0.5)

Linear		Quadratic (CE)		Quadratic	
E_1	E_∞	E_1	E_∞	E_1	E_∞
5.9956	10.1111	4.3885	4.7762	0.7784	1.9404

without taking care of the stochastic component of the problem, is not as high as taking the risky component into account. Indeed, accounting for curvature

Figure 6.1: Decision rule



Note: This graph was obtained for $\theta = -5$ and $\rho = -0.5$.

— *i.e.* moving from linear to quadratic (CE) — only leads to a 27% gain in terms of average error, where as taking risk into account permits to further enhance the accuracy by 82%, therefore yielding a 87% total gain! In others, increasing the order of the Taylor approximation, without taking into account the information carried by the moments of the distribution — ignoring the stochastic dimension of the problem — does not add that much in terms of accuracy. The main improvement can be precisely found in adding the higher order moments, as it yields a correction in the level of the rule, as can be seen from figure 1. As soon as the volatility of the shock explicitly appears in the rule then the rule shifts upward, thus getting closer to the true solution of the problem. Then, the average (maximal) approximation error decreases to 0.8% (1.9%) to be compared to the earlier 6% (10%). Therefore, most of the approximation error essentially lies in the level of the rule rather than in its curvature. Therefore, a higher curvature in the utility function necessitates an increase in the order of the moments of the distribution that should be taken

into account in the approximate solution.

There may however be situations where increasing the curvature constitutes *per se* a real improvement that contributes to reduce the approximation error. This is illustrated by considering the high persistence case (ρ=0.9). Indeed, as shown in table 6.1, increasing the order of the Taylor series expansion from 1 to 2 is not sufficient, *per se*, to solve the accuracy problem.

To sum up it appears that increasing the order of approximation and further considering information carried by the higher order moments of the distribution yields great improvements in the accuracy of the approximation, mainly because higher order moments enhance the ability of the approximation to match the level of the decision rule. However, problems still remain to be solved as the errors are still large for highly persistent economies. This can be easily explained if we consider the Taylor series expansion to the true decision rule, given by

$$y_t \simeq \sum_{i=0}^{\infty} \beta^i \exp(a_i) \left[\sum_{k=0}^{p} \frac{b_i^k}{k!} (x_t - \overline{x})^k \right]$$

Table 6.3 then reports the average and maximal errors induced by the Taylor expansion to the true rule. We only report these errors for cases where the previous analysis indicated a error greater than 1% for the O2 method. Table 6.3 clearly shows that approximation errors are large (greater than 1%) when a second order Taylor series expansion to the true rule is already not sufficient to produce an accurate representation to the analytical solution. For instance, in the $\theta = -10$ case, a "good" approximation of the true rule can be obtained only after a third order Taylor series expansion. This indicates that we should use at least a third order Taylor series expansion to increase significantly the accuracy of the approximation. This phenomenon is even more pronounced as we consider persistence. Let us focus on the case $\rho = 0.9$ for which approximation errors (see table 6.1) are huge — more than 15% in the O2 approximation. As can be seen from table 6.3, the second order Taylor series expansion to the true rule is very inaccurate as it produces maximal errors around 87%. An

Table 6.3: Taylor series expansion to the true solution

		Order of Taylor series expansion					
Case	Crit.	1	2	3	4	10	12
Benchmark	$E_1(v)$	0.01	0.00	0.00	0.00	0.00	0.00
	$E_\infty(v)$	0.03	0.00	0.00	0.00	0.00	0.00
β=0.99	$E_1(v)$	0.01	0.00	0.00	0.00	0.00	0.00
	$E_\infty(v)$	0.03	0.00	0.00	0.00	0.00	0.00
θ=-10	$E_1(v)$	0.49	0.02	0.00	0.00	0.00	0.00
	$E_\infty(v)$	1.60	0.09	0.00	0.00	0.00	0.00
θ=-5	$E_1(v)$	0.12	0.00	0.00	0.00	0.00	0.00
	$E_\infty(v)$	0.37	0.01	0.00	0.00	0.00	0.00
σ=0.1	$E_1(v)$	0.01	0.00	0.00	0.00	0.00	0.00
	$E_\infty(v)$	0.03	0.00	0.00	0.00	0.00	0.00
ρ=0.5	$E_1(v)$	0.63	0.03	0.00	0.00	0.00	0.00
	$E_\infty(v)$	2.05	0.14	0.01	0.00	0.00	0.00
ρ=0.9	$E_1(v)$	32.30	13.25	4.49	1.29	0.00	0.00
	$E_\infty(v)$	154.99	87.24	37.15	12.75	0.00	0.00

order of 12 is actually needed to generate an accurate approximation of the true rule. This hence explains the poor approximation — even using a global approach — we obtained in the $\rho = 0.9$ case, and indicates that a much higher corrected Taylor series expansion is required to increase accuracy. This thus does not invalidate the method but underlines one of its features, large orders can be necessary to get an accurate approximation.

Thus, error approximation results indicate that this perturbation procedure is accurate for most of the cases under study. But they also provide an explanation, and thus a solution, when the approximation is not accurate.

In this example, we only had to deal with an exogenous state variable, but it will be the case that we will have to deal with endogenous state variables such as the capital stock. Therefore, in the next section, we will deal with the optimal growth model.

6.2.3 The second order approximation to the optimal growth model

We now discuss the implementation of the perturbation method to this model. The first order optimality condition associated to this model, with a logarithmic utility function and a cobb douglas production function, write as

$$\beta E_t\left[\left(c_{t+1}^{-\sigma}\right)\left(\alpha\exp(a_{t+1})k_{t+1}^{\alpha-1}+1-\delta\right)\right]-c_t^{-\sigma}=0 \tag{6.19}$$

Finally, the law of motion of capital writes as

$$k_{t+1}-\exp(a_t)k_t^{\alpha}+c_t-(1-\delta)k_t \tag{6.20}$$

Therefore, the model to be solved consists of equations (6.19) and (6.20). Solving the model then amounts to find two functions $g(.)$ and $h(.)$ of the two predetermined variables k_t and a_t such that:

$$\begin{aligned} c_t &= g(k_t,a_t) \\ k_{t+1} &= h(k_t,a_t)=\exp(a_t)k_t^{\alpha}+(1-\delta)k_t-c_t \end{aligned}$$

Since $h(k_t,a_t)$ is entirely known once we have obtained $g(k_t,a_t)$, solving the mode reduces to find $g(k_t,a_t)$ that solves:

$$E_t\left[G(c_{t+1},k_{t+1},a_{t+1},c_t)\right]=0$$

Note that $c_{t+1}=g(k_{t+1},a_{t+1})=g(h(k_t,a_t),a_{t+1})=f(k_t,a_t,\varepsilon_{t+1})$ so that the later equation rewrites as

$$E_t\left[G(f(k_t,a_t,\sigma),h(k_t,a_t),\rho a_{t+1}+(1-\rho)\bar{a}+\sigma\varepsilon_{t+1},g(k_t,a_t))\right]=0$$

or

$$E_t\left[F(k_t,a_t,\sigma)\right]=0 \tag{6.21}$$

We now present the perturbation method for the quadratic case. Taking the second order Taylor expansion of (6.21) around $(\tilde{k}, \tilde{a}, 0)$ yields

$$\begin{aligned} E_t\Big[& F(\tilde{k},\tilde{a},0) + F_k(\tilde{k},\tilde{a},0)\hat{k}_t + F_a(\tilde{k},\tilde{a},0)\hat{a}_t + F_\sigma(\tilde{k},\tilde{a},0)\hat{\sigma} \\ & + \frac{1}{2}F_{kk}(\tilde{k},\tilde{a},0)\hat{k}_t^2 + \frac{1}{2}F_{aa}(\tilde{k},\tilde{a},0)\hat{a}_t^2 + \frac{1}{2}F_{\sigma\sigma}(\tilde{k},\tilde{a},0)\hat{\sigma}^2 \\ & + F_{ka}(\tilde{k},\tilde{a},0)\hat{k}_t\hat{a}_t + F_{k\sigma}(\tilde{k},\tilde{a},0)\hat{k}_t\hat{\sigma} + F_{a\sigma}(\tilde{k},\tilde{a},0)\hat{a}_t\hat{\sigma}\Big] = 0 \end{aligned} \quad (6.22)$$

Since k_t and a_t are perfectly known to the household when she takes her decisions, and since σ is a constant, expectations can be dropped off in the previous equation, such that (6.22) reduces to

$$\begin{aligned} & F(\tilde{k},\tilde{a},0) + F_k(\tilde{k},\tilde{a},0)\hat{k}_t + F_a(\tilde{k},\tilde{a},0)\hat{a}_t + F_\sigma(\tilde{k},\tilde{a},0)\hat{\sigma} \\ & + \frac{1}{2}F_{kk}(\tilde{k},\tilde{a},0)\hat{k}_t^2 + \frac{1}{2}F_{aa}(\tilde{k},\tilde{a},0)\hat{a}_t^2 + \frac{1}{2}F_{\sigma\sigma}(\tilde{k},\tilde{a},0)\hat{\sigma}^2 \\ & + F_{ka}(\tilde{k},\tilde{a},0)\hat{k}_t\hat{a}_t + F_{k\sigma}(\tilde{k},\tilde{a},0)\hat{k}_t\hat{\sigma} + F_{a\sigma}(\tilde{k},\tilde{a},0)\hat{a}_t\hat{\sigma} = 0 \end{aligned} \quad (6.23)$$

Likewise, $g(.)$ is approximated by

$$\begin{aligned} g(k_t, a_t, \sigma) \simeq & \; g(\tilde{k},\tilde{a},0) + g_k(\tilde{k},\tilde{a},0)\hat{k}_t + g_a(\tilde{k},\tilde{a},0)\hat{a}_t + g_\sigma(\tilde{k},\tilde{a},0)\hat{\sigma} \\ & + \frac{1}{2}g_{kk}(\tilde{k},\tilde{a},0)\hat{k}_t^2 + \frac{1}{2}g_{aa}(\tilde{k},\tilde{a},0)\hat{a}_t^2 + \frac{1}{2}g_{\sigma\sigma}(\tilde{k},\tilde{a},0)\hat{\sigma}^2 \\ & + g_{ka}(\tilde{k},\tilde{a},0)\hat{k}_t\hat{a}_t + g_{k\sigma}(\tilde{k},\tilde{a},0)\hat{k}_t\hat{\sigma} + g_{a\sigma}(\tilde{k},\tilde{a},0)\hat{a}_t\hat{\sigma} \end{aligned} \quad (6.24)$$

Note however that the general approach to the problem has enabled us to establish formally that both $g_\sigma(\tilde{k},\tilde{a},0)$, $g_{k\sigma}(\tilde{k},\tilde{a},0)$ and $g_{a\sigma}(\tilde{k},\tilde{a},0)$ should be equal to 0. such that we just have to find $g(\tilde{k},\tilde{a},0)$, $g_k(\tilde{k},\tilde{a},0)$, $g_a(\tilde{k},\tilde{a},0)$, $g_{kk}(\tilde{k},\tilde{a},0)$, $g_{ka}(\tilde{k},\tilde{a},0)$, $g_{aa}(\tilde{k},\tilde{a},0)$ and $g_{\sigma\sigma}(\tilde{k},\tilde{a},0)$. Identifying each term

amounts to solve the system:[3]

$$\begin{aligned} F(\tilde{k},\tilde{a},0) &= 0 \\ F_k(\tilde{k},\tilde{a},0) &= 0 \\ F_a(\tilde{k},\tilde{a},0) &= 0 \\ F_{kk}(\tilde{k},\tilde{a},0) &= 0 \\ F_{aa}(\tilde{k},\tilde{a},0) &= 0 \\ F_{\sigma\sigma}(\tilde{k},\tilde{a},0) &= 0 \\ F_{ka}(\tilde{k},\tilde{a},0) &= 0 \end{aligned}$$

The first of these equations actually defines the steady state, where as the next two equations actually amounts to solve the linearized system, as we have shown in the theoretical section. This system can be solved numerically using different packages.[4] The following codes illustrate the implementation of the method for $\alpha = 0.3$, $\delta = 0.1$, $\beta = 0.95$, $\sigma = 2.5$, $\rho = 0.9$ and a volatility of 0.1.

Matlab Code: Perturbation Method (the OGM)

```
clear all;
nx  = 2;    % # backward variables (k,a)
ny  = 1;    % # forward and static variables (c)
ne  = 1;    % # shocks (a)

alpha   = 0.3;          % capital elasticity of output
sigma   = 2.5;          % parameter of the utility function
beta    = 0.95;         % discount factor
delta   = 0.1;          % depreciation rate
ab      = 0;            % average of the log of technology shock
rhoa    = 0.9;          % persistence of the technology shock
s2      = 1;
eta1    = 0.1^2;        % volatility of the shock
ETA     = [0;eta1];
```

[3]The exact form of this system is given in appendix.

[4]Schmitt-Grohe and Uribe [2001] propose their own package, Chris Sims has also one package that you can download from his webpage: `http://www.princeton.edu/ sims/`, in these notes I am using a package that M. Juillard and I designed which is based on Schmitt-Grohe and Uribe [2001] but that does not require additional matlab toolboxes like the symbolic toolbox.

```
%
% Steady state
%
ksy     =alpha*beta/(1-beta*(1-delta));
ysk     =(1-beta*(1-delta))/(alpha*beta);
ys      =ksy^(alpha/(1-alpha));
ks      =ksy*ys;
cs      =ys*(1-delta*ksy);
%
% Solving the model
%
param   = [beta sigma alpha delta rhoa as]; % vector of parameters
xs      = [ks as cs ks as cs]';             % deterministic steady state
[J]     = diffext('model',xs,[],param);     % Jacobian matrix of the model
[H]     = hessext('model',xs,[],param);     % Hessian matric of the model

[Gx,Hx,Gxx,Hxx,Gss,Hss]=solve(xs,J,H,ETA,nx,ny,ne);
%
% Simulating the economy (only once)
%
long    = 120;
tronc   = 100;
slong   = long+tronc;
T       = tronc+1:slong;
sim     = 1;
seed    = 1;
e       = randn(slong,ne)*s2;
S1      = zeros(nx,slong);
S2      = zeros(nx,slong);
X1      = zeros(ny,slong);
X2      = zeros(ny,slong);
S1(:,1) = ETA*e(:,1);
S2(:,1) = ETA*e(:,1);
tmp     = S2(:,1)*S2(:,1)';
X1(:,1) = Gx*S1(:,1);
X2(:,1) = X1(:,1)+0.5*Gxx*tmp(:);
for i=2:slong
   S1(:,i)=Hx*S1(:,i-1)+ETA*e(:,i);
   X1(:,i)=Gx*S1(:,i);
   S2(:,i)=S1(:,i)+0.5*Hxx*tmp(:)+0.5*Hss;
   tmp=S2(:,i)*S2(:,i)';
   X2(:,i)=(Gs+0.5*Gss*s2+X1(:,i)+0.5*Gxx*tmp(:))';
   X1(:,i)=(Gs+X1(:,i))';
end;
```

MATLAB CODE: THE MODEL

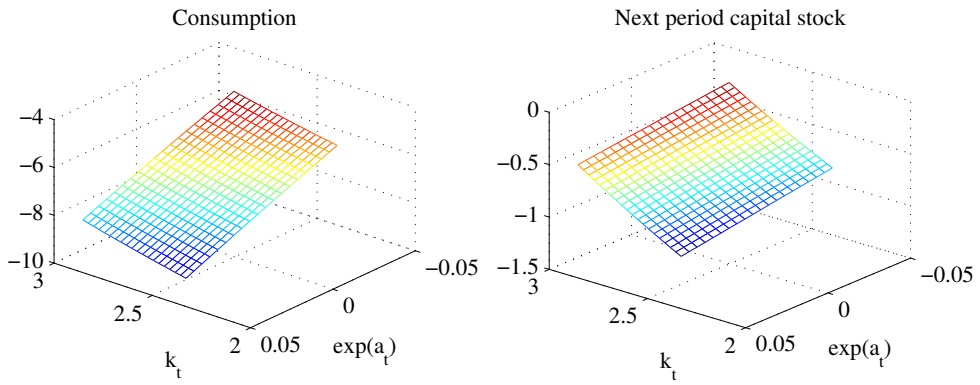
```
function eq=model(xx,param);
%
% parameters
%
beta    = param(1);
sigma   = param(2);
alpha   = param(3);
delta   = param(4);
ra      = param(5);
ab      = param(6);
%
% variables (leads and lags)
%
kp      = xx(1); % k(t+1)
ap      = xx(2); % a(t+1)
cp      = xx(3); % c(t+1)
k       = xx(4); % k(t)
a       = xx(5); % a(t)
c       = xx(6); % c(t)
%
% Input the model as we write it on a sheet of paper
%
eq      = zeros(3,1);
%
% Backward variables
%
% k' a' c' k a c
% 1  2  3  4 5 6
eq(1)   = kp-exp(a)*k^alpha+c-(1-delta)*k;
eq(2)   = ap-ra*a-(1-ra)*ab;
%
% Forward variables
%
eq(3)   = c^(-sigma)-beta*(cp^(-sigma))*(alpha*exp(ap)*(kp^(alpha-1))+1-delta);
```

Figure (6.2) reports the relative differences in the decision rules for consumption and capital between the quadratic and the linear approximation, computed as

$$100 \times \frac{g^{quad} - g^{lin}}{g^{lin}}$$

As it appears clearly in the figure, taking into account risk induces a drop in the level of consumption between 5 and 10% relative to the linear approximation. This actually just reflects the precautionary motive that is at work

Figure 6.2: Differences between Linear and Quadratic approximations



in this type of model. Since the agent is risk averse, she increase her savings in order to insure herself against shocks, therefore lowering her current expenditures. This also affects current investment to a lesser extent therefore translating into lower capital accumulation.

6.3 Some words of caution

- The perturbation method is and remains a local approximation. Therefore, even in in practice it may handle larger shocks than the now conventional (log–)linear approximation, it cannot be used to study the implications of big structural shocks such as a tax reform, unless this reform is marginal.

- By construction, this method requires the model to be differentiable, which forbids the analysis of modes with binding constraints.

- The quadratic approximation may sometime induce strange behaviors due to the quadratic term. In order to understand the problem, let us focus on a simplistic one dimensional model for which the the quadratic approximation of the decision rule for the state variable would be

$$x_{t+1} - x^\star = \alpha_0(x_t - x^\star) + \alpha_1(x_t - x^\star)^2$$

which may be rewritten as

$$\widehat{x}_{t+1} = \alpha_0 \widehat{x}_t (1 - \alpha_2 \widehat{x}_t)$$

One can recognize the so–called logistic map that lies at the core of a lot of chaotic systems. Such that we may encounter cases for which the model is known to be locally stable, which can be checked taking a linear approximation. But the quadratic approximation can lead to chaotic dynamics or at least unattended behavior. (try for example to compute the IRF of capital to a 1% increase when a second order approximation is used with $\sigma = 33.7145$!). Note that this does not totally question the relevance of the approach but rather reveals a problem of accuracy which calls for either other methods, or higher order approximations.

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