

Technical Appendix: Imperfect information and the business cycle *

Fabrice Collard[†] Harris Dellas[‡] Frank Smets[§]

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[†]School of Economics, The University of Adelaide, SA 5005 Australia. Tel: (+61) (0)8-8303-4928 Fax: (+61) (0)8- 8223 1460 URL: email: fabrice.collard@gmail.com, Homepage:<http://fabcol.free.fr>

[‡]Department of Economics, University of Bern, CEPR. Address: VWI, Schanzeneckstrasse 1, CH 3012 Bern, Switzerland. Tel: (41) 31-631-3989, Fax: (41) 31-631-3992, email: harris.dellas@vwi.unibe.ch, Homepage: <http://www.vwi.unibe.ch/amakro/dellas.htm>

[§]Frank Smets: European Central Bank, CEPR and Ghent University, Kaiderstrasse 29 D-60311 Frankfurt am Main, Germany, Tel: (49)-69-1344 8782 Fax: (49)-69-1344 8553 e-mail: frank.smets@ecb.int

1 Robustness Analysis

Table 1: Diffuse Priors

| Param. | Type | Param 1 | Param 2 | 95% HPDI |
|---------------|----------|---------|---------|---------------|
| ϑ | Uniform | 0.00 | 1.00 | [0.025;0.975] |
| ξ | Uniform | 0.00 | 1.00 | [0.025;0.975] |
| φ | Uniform | 0.00 | 1.00 | [0.025;0.975] |
| r^* | Uniform | 0.00 | 4.00 | [0.10;3.90] |
| π^* | Uniform | 0.00 | 4.00 | [0.10;3.90] |
| ρ_r | Uniform | 0.00 | 1.00 | [0.025;0.975] |
| α_π | Normal | 1.50 | 0.50 | [0.52;2.47] |
| α_y | Normal | 0.125 | 0.05 | [0.027;0.222] |
| ρ_a | Uniform | 0.00 | 1.00 | [0.025;0.975] |
| ρ_χ | Uniform | 0.00 | 1.00 | [0.025;0.975] |
| ρ_π | Uniform | 0.50 | 0.50 | [0.025;0.975] |
| σ_a | Invgamma | 0.20 | 4.00 | [0.10;0.38] |
| σ_χ | Invgamma | 0.20 | 4.00 | [0.10;0.38] |
| σ_r | Invgamma | 0.20 | 4.00 | [0.10;0.38] |
| σ_π | Invgamma | 0.20 | 4.00 | [0.10;0.38] |
| σ_ν | Invgamma | 0.20 | 4.00 | [0.10;0.38] |
| η_y | Invgamma | 0.20 | 4.00 | [0.10;0.38] |
| η_π | Invgamma | 0.20 | 4.00 | [0.10;0.38] |

Note: The parameters are distributed independently from each other.
^a 95-percent highest probability density (HPD) credible intervals (see ?, p.57). The Param 1 and Param 2 report the lower and upper bounds for Uniform distributions, the mean and the standard deviation for the Normal distributions. They report the s and ν parameters of the inverse gamma distribution, where $f(\sigma|s, \nu) \propto \sigma^{-(1+\nu)} \exp(-\nu s^2 / 2\sigma^2)$.

Table 2: Moments: Data linearly detrended

| | Data | (1) | (2) | (3) | (4) | (5) |
|----------------|-------|---------------|---------------|---------------|---------------|--------------|
| σ_y | 3.37 | 5.87 | 4.09 | 6.11 | 6.76 | 2.50 |
| | | [3.11,9.86] | [2.49,5.94] | [2.95,10.39] | [3.13,12.08] | [2.00,3.09] |
| σ_π | 0.62 | 0.55 | 0.70 | 0.54 | 0.62 | 0.51 |
| | | [0.45,0.68] | [0.57,0.85] | [0.44,0.67] | [0.48,0.81] | [0.41,0.61] |
| σ_R | 0.79 | 0.74 | 0.89 | 0.72 | 0.93 | 0.60 |
| | | [0.55,1.00] | [0.70,1.10] | [0.53,0.97] | [0.63,1.33] | [0.48,0.74] |
| $\rho(\pi, y)$ | -0.27 | -0.18 | 0.08 | -0.17 | -0.06 | -0.01 |
| | | [-0.35,-0.04] | [-0.26,0.37] | [-0.32,-0.03] | [-0.10,-0.02] | [-0.15,0.15] |
| $\rho(R, y)$ | -0.39 | -0.32 | -0.36 | -0.31 | -0.13 | 0.03 |
| | | [-0.52,-0.12] | [-0.65,-0.05] | [-0.50,-0.12] | [-0.23,-0.04] | [-0.17,0.27] |
| $\rho_y(1)$ | 0.97 | 0.97 | 0.97 | 0.97 | 0.99 | 0.93 |
| | | [0.94,1.00] | [0.95,0.99] | [0.94,1.00] | [0.97,1.00] | [0.90,0.96] |
| $\rho_\pi(1)$ | 0.87 | 0.69 | 0.88 | 0.70 | 0.81 | 0.76 |
| | | [0.56,0.82] | [0.83,0.92] | [0.57,0.83] | [0.71,0.90] | [0.66,0.85] |
| $\rho_R(1)$ | 0.93 | 0.90 | 0.94 | 0.90 | 0.93 | 0.89 |
| | | [0.85,0.96] | [0.91,0.96] | [0.84,0.96] | [0.89,0.98] | [0.85,0.94] |
| $\rho_y(2)$ | 0.92 | 0.95 | 0.93 | 0.95 | 0.97 | 0.86 |
| | | [0.89,0.99] | [0.87,0.98] | [0.89,1.00] | [0.94,1.00] | [0.81,0.91] |
| $\rho_\pi(2)$ | 0.83 | 0.56 | 0.75 | 0.57 | 0.74 | 0.71 |
| | | [0.39,0.74] | [0.67,0.84] | [0.39,0.73] | [0.61,0.87] | [0.60,0.80] |
| $\rho_R(2)$ | 0.84 | 0.83 | 0.86 | 0.82 | 0.86 | 0.79 |
| | | [0.74,0.92] | [0.80,0.91] | [0.73,0.91] | [0.77,0.95] | [0.71,0.87] |
| $\rho_y(4)$ | 0.79 | 0.91 | 0.82 | 0.91 | 0.95 | 0.76 |
| | | [0.81,0.99] | [0.70,0.93] | [0.82,0.99] | [0.89,1.00] | [0.67,0.84] |
| $\rho_\pi(4)$ | 0.79 | 0.44 | 0.54 | 0.44 | 0.64 | 0.62 |
| | | [0.25,0.64] | [0.40,0.69] | [0.26,0.65] | [0.46,0.80] | [0.51,0.73] |
| $\rho_R(4)$ | 0.69 | 0.70 | 0.68 | 0.69 | 0.74 | 0.61 |
| | | [0.58,0.86] | [0.55,0.80] | [0.55,0.84] | [0.59,0.90] | [0.50,0.73] |

Note: (1): Baseline NK, (2): Hybrid NK (Backward Indexation, Real Rigidities), (3): Imperfect Info. Temporary vs Permanent Shocks, (4): Imperfect Info., Cogley–Sbordone, (5): Imperfect Info., Noisy Signals.

2 Detailed Tables, 1966-2002

Table 3: Posteriors – Perfect Info, Forward NK

| Param. | Mode | Mean | Median | Std. Dev. | 95% HPDI |
|-------------------------------|------|------|--------|-----------|---------------|
| ξ | 0.48 | 0.49 | 0.50 | 0.08 | [0.33, 0.64] |
| φ | 0.25 | 0.26 | 0.26 | 0.11 | [0.04, 0.48] |
| r^* | 0.63 | 0.61 | 0.62 | 0.14 | [0.31, 0.86] |
| π^* | 0.93 | 0.90 | 0.92 | 0.18 | [0.53, 1.24] |
| ρ_r | 0.68 | 0.67 | 0.68 | 0.04 | [0.60, 0.75] |
| α_π | 1.54 | 1.61 | 1.60 | 0.16 | [1.29, 1.90] |
| α_y | 0.13 | 0.13 | 0.13 | 0.05 | [0.04, 0.23] |
| ρ_a | 0.98 | 0.98 | 0.98 | 0.01 | [0.96, 1.00] |
| ρ_χ | 0.91 | 0.91 | 0.91 | 0.03 | [0.85, 0.97] |
| σ_a | 0.24 | 0.27 | 0.26 | 0.08 | [0.14, 0.44] |
| σ_χ | 0.15 | 0.16 | 0.16 | 0.03 | [0.11, 0.22] |
| σ_r | 0.34 | 0.36 | 0.35 | 0.03 | [0.30, 0.41] |
| σ_ν | 0.14 | 0.14 | 0.14 | 0.02 | [0.10, 0.19] |
| Average log marginal density: | | | | | -323.472 |

Table 4: Posteriors – Perfect Info, Hybrid NK

| Param. | Mode | Mean | Median | Std. Dev. | 95% HPDI |
|-------------------------------|------|------|--------|-----------|---------------|
| θ | 0.94 | 0.94 | 0.94 | 0.02 | [0.90, 0.97] |
| ξ | 0.05 | 0.08 | 0.08 | 0.03 | [0.03, 0.14] |
| φ | 0.26 | 0.29 | 0.29 | 0.11 | [0.07, 0.51] |
| r^* | 0.66 | 0.66 | 0.66 | 0.15 | [0.36, 0.97] |
| π^* | 1.03 | 1.01 | 1.01 | 0.16 | [0.69, 1.33] |
| ρ_r | 0.86 | 0.84 | 0.84 | 0.04 | [0.77, 0.91] |
| α_π | 1.40 | 1.40 | 1.38 | 0.18 | [1.06, 1.75] |
| α_y | 0.00 | 0.04 | 0.03 | 0.04 | [0.00, 0.12] |
| ρ_a | 0.02 | 0.07 | 0.06 | 0.05 | [0.00, 0.18] |
| ρ_χ | 0.23 | 0.26 | 0.26 | 0.08 | [0.09, 0.42] |
| σ_a | 0.12 | 0.11 | 0.11 | 0.02 | [0.08, 0.14] |
| σ_χ | 0.61 | 0.62 | 0.62 | 0.07 | [0.48, 0.77] |
| σ_r | 0.26 | 0.26 | 0.26 | 0.02 | [0.23, 0.29] |
| σ_ν | 0.12 | 0.15 | 0.15 | 0.02 | [0.11, 0.18] |
| Average log marginal density: | | | | | -267.388 |

Table 5: Posteriors – Imperfect Info, Pers. vs Temp.

| Param. | Mode | Mean | Median | Std. Dev. | 95% HPDI |
|-------------------------------|------|------|--------|-----------|---------------|
| ξ | 0.47 | 0.49 | 0.49 | 0.08 | [0.34, 0.64] |
| φ | 0.25 | 0.27 | 0.26 | 0.11 | [0.05, 0.49] |
| r^* | 0.64 | 0.62 | 0.63 | 0.13 | [0.34, 0.87] |
| π^* | 0.93 | 0.91 | 0.93 | 0.17 | [0.57, 1.26] |
| ρ_r | 0.68 | 0.67 | 0.68 | 0.04 | [0.60, 0.74] |
| α_π | 1.50 | 1.58 | 1.58 | 0.15 | [1.31, 1.90] |
| α_y | 0.15 | 0.15 | 0.15 | 0.05 | [0.05, 0.24] |
| ρ_a | 0.99 | 0.98 | 0.98 | 0.01 | [0.96, 1.00] |
| ρ_χ | 0.91 | 0.91 | 0.91 | 0.03 | [0.85, 0.97] |
| σ_a | 0.21 | 0.25 | 0.24 | 0.08 | [0.12, 0.40] |
| σ_χ | 0.16 | 0.17 | 0.17 | 0.03 | [0.12, 0.23] |
| σ_r | 0.34 | 0.35 | 0.35 | 0.03 | [0.30, 0.41] |
| σ_ν | 0.16 | 0.16 | 0.16 | 0.03 | [0.11, 0.22] |
| Average log marginal density: | | | | | -321.101 |

Table 6: Posteriors – Imperfect info, Inflation target shock

| Param. | Mode | Mean | Median | Std. Dev. | 95% HPDI |
|-------------------------------|------|------|--------|-----------|---------------|
| ξ | 0.70 | 0.68 | 0.69 | 0.08 | [0.51, 0.83] |
| φ | 0.24 | 0.24 | 0.24 | 0.12 | [0.02, 0.46] |
| r^* | 0.59 | 0.56 | 0.58 | 0.21 | [0.11, 0.92] |
| π^* | 0.89 | 0.86 | 0.88 | 0.24 | [0.34, 1.30] |
| ρ_r | 0.20 | 0.21 | 0.20 | 0.10 | [0.02, 0.39] |
| α_π | 2.33 | 2.37 | 2.37 | 0.17 | [2.05, 2.70] |
| α_y | 0.13 | 0.13 | 0.13 | 0.05 | [0.03, 0.22] |
| ρ_a | 0.99 | 0.99 | 0.99 | 0.01 | [0.97, 1.00] |
| ρ_χ | 0.92 | 0.92 | 0.92 | 0.03 | [0.85, 0.98] |
| ρ_π | 0.93 | 0.92 | 0.93 | 0.03 | [0.86, 0.98] |
| σ_a | 0.78 | 0.81 | 0.77 | 0.28 | [0.35, 1.38] |
| σ_χ | 0.19 | 0.20 | 0.20 | 0.02 | [0.16, 0.24] |
| σ_r | 0.49 | 0.49 | 0.49 | 0.07 | [0.37, 0.63] |
| σ_π | 0.11 | 0.11 | 0.11 | 0.02 | [0.08, 0.15] |
| Average log marginal density: | | | | | -296.147 |

Table 7: Posteriors – Imperfect info, Measurement errors

| Param. | Mode | Mean | Median | Std. Dev. | 95% HPDI |
|-------------------------------|------|------|--------|-----------|----------------|
| ξ | 0.23 | 0.22 | 0.22 | 0.04 | [0.14, 0.31] |
| φ | 0.26 | 0.28 | 0.28 | 0.11 | [0.06, 0.50] |
| r^* | 0.64 | 0.62 | 0.62 | 0.20 | [0.22, 1.02] |
| π^* | 0.98 | 0.97 | 0.97 | 0.20 | [0.57, 1.34] |
| ρ_r | 0.27 | 0.27 | 0.28 | 0.07 | [0.14, 0.42] |
| α_π | 1.62 | 1.64 | 1.64 | 0.23 | [1.19, 2.10] |
| α_y | 0.19 | 0.20 | 0.20 | 0.04 | [0.11, 0.28] |
| ρ_a | 0.94 | 0.95 | 0.95 | 0.01 | [0.92, 0.97] |
| ρ_χ | 0.85 | 0.86 | 0.86 | 0.03 | [0.80, 0.92] |
| σ_a | 0.11 | 0.12 | 0.12 | 0.02 | [0.09, 0.16] |
| σ_χ | 0.28 | 0.29 | 0.28 | 0.04 | [0.21, 0.37] |
| σ_r | 0.12 | 0.13 | 0.13 | 0.02 | [0.09, 0.16] |
| σ_ν | 0.22 | 0.21 | 0.21 | 0.02 | [0.18, 0.25] |
| η_y | 0.21 | 0.28 | 0.25 | 0.12 | [0.10, 0.52] |
| η_π | 4.34 | 6.72 | 6.10 | 3.02 | [2.18, 13.15] |
| Average log marginal density: | | | | | -264.729 |

2.1 Detailed Tables, More diffuse priors

Table 8: Posteriors – Perfect Info, Forward NK

| Param. | Mode | Mean | Median | Std. Dev. | 95% HPDI |
|-------------------------------|------|------|--------|-----------|-------------|
| ξ | 0.56 | 0.58 | 0.59 | 0.12 | [0.35,0.77] |
| φ | 0.40 | 0.48 | 0.46 | 0.28 | [0.01,0.93] |
| r^* | 0.66 | 0.65 | 0.66 | 0.14 | [0.36,0.94] |
| π^* | 0.96 | 0.96 | 0.96 | 0.18 | [0.59,1.32] |
| ρ_r | 0.68 | 0.68 | 0.68 | 0.04 | [0.60,0.75] |
| α_π | 1.55 | 1.64 | 1.64 | 0.18 | [1.29,2.02] |
| α_y | 0.14 | 0.15 | 0.14 | 0.07 | [0.02,0.28] |
| ρ_a | 0.99 | 0.98 | 0.98 | 0.01 | [0.96,1.00] |
| ρ_χ | 0.92 | 0.92 | 0.92 | 0.03 | [0.86,0.98] |
| σ_a | 0.26 | 0.29 | 0.28 | 0.09 | [0.14,0.48] |
| σ_χ | 0.15 | 0.16 | 0.16 | 0.03 | [0.11,0.21] |
| σ_r | 0.35 | 0.36 | 0.36 | 0.03 | [0.30,0.42] |
| σ_ν | 0.14 | 0.15 | 0.14 | 0.03 | [0.10,0.19] |
| Average log marginal density: | | | | | -313.883 |

Table 9: Posteriors – Perfect Info, Hybrid NK

| Param. | Mode | Mean | Median | Std. Dev. | 95% HPDI |
|-------------------------------|------|------|--------|-----------|---------------|
| θ | 0.93 | 0.94 | 0.94 | 0.02 | [0.90, 0.98] |
| ξ | 0.08 | 0.09 | 0.09 | 0.04 | [0.03, 0.17] |
| φ | 1.00 | 0.56 | 0.58 | 0.26 | [0.12, 1.00] |
| r^* | 0.66 | 0.70 | 0.70 | 0.15 | [0.40, 0.99] |
| π^* | 1.15 | 1.03 | 1.03 | 0.16 | [0.73, 1.34] |
| ρ_r | 0.85 | 0.85 | 0.85 | 0.04 | [0.77, 0.92] |
| α_π | 1.26 | 1.40 | 1.37 | 0.22 | [1.00, 1.83] |
| α_y | 0.00 | 0.03 | 0.02 | 0.03 | [0.00, 0.09] |
| ρ_a | 0.00 | 0.05 | 0.04 | 0.05 | [0.00, 0.15] |
| ρ_χ | 0.42 | 0.24 | 0.24 | 0.08 | [0.07, 0.40] |
| σ_a | 0.08 | 0.11 | 0.11 | 0.02 | [0.08, 0.14] |
| σ_χ | 0.47 | 0.63 | 0.63 | 0.07 | [0.49, 0.78] |
| σ_r | 0.26 | 0.26 | 0.26 | 0.02 | [0.23, 0.30] |
| σ_ν | 0.15 | 0.14 | 0.14 | 0.02 | [0.10, 0.18] |
| Average log marginal density: | | | | | -258.469 |

Table 10: Posteriors – Imperfect Info, Pers. vs Temp.

| Param. | Mode | Mean | Median | Std. Dev. | 95% HPDI |
|-------------------------------|------|------|--------|-----------|---------------|
| ξ | 0.60 | 0.57 | 0.59 | 0.12 | [0.34, 0.77] |
| φ | 0.60 | 0.51 | 0.51 | 0.27 | [0.06, 0.98] |
| r^* | 0.66 | 0.66 | 0.67 | 0.14 | [0.37, 0.93] |
| π^* | 0.96 | 0.97 | 0.98 | 0.18 | [0.60, 1.31] |
| ρ_r | 0.68 | 0.68 | 0.68 | 0.04 | [0.60, 0.76] |
| α_π | 1.48 | 1.59 | 1.58 | 0.19 | [1.20, 1.96] |
| α_y | 0.17 | 0.17 | 0.17 | 0.07 | [0.04, 0.31] |
| ρ_a | 0.99 | 0.99 | 0.99 | 0.01 | [0.97, 1.00] |
| ρ_χ | 0.91 | 0.91 | 0.91 | 0.03 | [0.85, 0.97] |
| σ_a | 0.21 | 0.25 | 0.24 | 0.09 | [0.11, 0.41] |
| σ_χ | 0.15 | 0.17 | 0.16 | 0.03 | [0.11, 0.23] |
| σ_r | 0.34 | 0.36 | 0.35 | 0.03 | [0.30, 0.42] |
| σ_ν | 0.16 | 0.16 | 0.16 | 0.03 | [0.11, 0.22] |
| Average log marginal density: | | | | | -317.726 |

Table 11: Posteriors – Imperfect info, Inflation target shock

| Param. | Mode | Mean | Median | Std. Dev. | 95% HPDI |
|-------------------------------|------|------|--------|-----------|---------------|
| ξ | 0.70 | 0.76 | 0.78 | 0.10 | [0.56, 0.91] |
| φ | 0.14 | 0.38 | 0.36 | 0.23 | [0.00, 0.74] |
| r^* | 0.65 | 0.68 | 0.67 | 0.25 | [0.15, 1.16] |
| π^* | 0.95 | 0.94 | 0.94 | 0.27 | [0.39, 1.47] |
| ρ_r | 0.08 | 0.18 | 0.17 | 0.11 | [0.00, 0.37] |
| α_π | 2.90 | 3.03 | 3.01 | 0.29 | [2.47, 3.61] |
| α_y | 0.12 | 0.14 | 0.14 | 0.07 | [0.02, 0.26] |
| ρ_a | 1.00 | 0.99 | 0.99 | 0.01 | [0.97, 1.00] |
| ρ_χ | 0.92 | 0.92 | 0.93 | 0.03 | [0.86, 0.98] |
| ρ_π | 0.94 | 0.94 | 0.94 | 0.03 | [0.89, 0.99] |
| σ_a | 1.14 | 1.04 | 1.00 | 0.33 | [0.44, 1.71] |
| σ_χ | 0.20 | 0.21 | 0.21 | 0.02 | [0.18, 0.25] |
| σ_r | 0.64 | 0.61 | 0.60 | 0.09 | [0.43, 0.78] |
| σ_π | 0.11 | 0.11 | 0.11 | 0.02 | [0.08, 0.14] |
| Average log marginal density: | | | | | -276.183 |

Table 12: Posteriors – Imperfect info, Measurement errors

| Param. | Mode | Mean | Median | Std. Dev. | 95% HPDI |
|-------------------------------|------|------|--------|-----------|----------------|
| ξ | 0.29 | 0.30 | 0.30 | 0.07 | [0.15, 0.42] |
| φ | 0.50 | 0.60 | 0.64 | 0.27 | [0.11, 1.00] |
| r^* | 0.69 | 0.68 | 0.68 | 0.21 | [0.28, 1.11] |
| π^* | 0.98 | 0.97 | 0.97 | 0.19 | [0.59, 1.34] |
| ρ_r | 0.28 | 0.29 | 0.29 | 0.08 | [0.14, 0.44] |
| α_π | 1.61 | 1.76 | 1.74 | 0.39 | [1.00, 2.45] |
| α_y | 0.23 | 0.23 | 0.23 | 0.06 | [0.11, 0.35] |
| ρ_a | 0.95 | 0.95 | 0.95 | 0.01 | [0.92, 0.97] |
| ρ_χ | 0.85 | 0.86 | 0.86 | 0.03 | [0.80, 0.93] |
| σ_a | 0.11 | 0.12 | 0.12 | 0.02 | [0.09, 0.16] |
| σ_χ | 0.28 | 0.30 | 0.29 | 0.04 | [0.22, 0.38] |
| σ_r | 0.12 | 0.13 | 0.13 | 0.02 | [0.09, 0.17] |
| σ_ν | 0.22 | 0.21 | 0.21 | 0.02 | [0.18, 0.25] |
| η_y | 0.20 | 0.28 | 0.24 | 0.13 | [0.11, 0.53] |
| η_π | 4.33 | 6.39 | 5.81 | 2.85 | [2.08, 12.47] |
| Average log marginal density: | | | | | -255.363 |

2.2 Detailed Tables, Alternative Specifications

Table 13: Posteriors – Perfect Info, Hybrid NK (**Partial indexation to the inflation target**)

| Param. | Mode | Mean | Median | Std. Dev. | 95% HPDI |
|-------------------------------|------|------|--------|-----------|-------------|
| θ | 0.94 | 0.92 | 0.93 | 0.02 | [0.88,0.97] |
| ξ | 0.02 | 0.07 | 0.07 | 0.02 | [0.03,0.12] |
| γ | 0.85 | 0.88 | 0.89 | 0.05 | [0.79,0.97] |
| φ | 0.26 | 0.28 | 0.28 | 0.11 | [0.07,0.51] |
| r^* | 0.67 | 0.67 | 0.67 | 0.12 | [0.42,0.92] |
| π^* | 1.02 | 1.02 | 1.02 | 0.13 | [0.75,1.27] |
| ρ_r | 0.84 | 0.76 | 0.76 | 0.06 | [0.64,0.86] |
| α_π | 1.21 | 1.21 | 1.19 | 0.13 | [1.01,1.47] |
| α_y | 0.00 | 0.02 | 0.02 | 0.01 | [0.00,0.03] |
| ρ_a | 0.03 | 0.08 | 0.07 | 0.06 | [0.00,0.19] |
| ρ_χ | 0.24 | 0.28 | 0.28 | 0.09 | [0.12,0.45] |
| ρ_π | 0.02 | 0.09 | 0.08 | 0.07 | [0.00,0.22] |
| σ_a | 0.12 | 0.12 | 0.12 | 0.02 | [0.09,0.16] |
| σ_χ | 0.61 | 0.60 | 0.59 | 0.07 | [0.45,0.74] |
| σ_r | 0.26 | 0.23 | 0.23 | 0.03 | [0.17,0.29] |
| σ_π | 0.12 | 0.12 | 0.12 | 0.02 | [0.09,0.16] |
| Average log marginal density: | | | | | -278.186 |

Table 14: Posteriors – Imperfect info, Measurement errors (R and π are perfectly observable)

| Param. | Mode | Mean | Median | Std. Dev. | 95% HPDI |
|-------------------------------|------|------|--------|-----------|-------------|
| ξ | 0.46 | 0.48 | 0.48 | 0.08 | [0.34,0.64] |
| φ | 0.25 | 0.27 | 0.27 | 0.12 | [0.05,0.50] |
| r^* | 0.64 | 0.62 | 0.63 | 0.13 | [0.37,0.89] |
| π^* | 0.93 | 0.92 | 0.94 | 0.17 | [0.54,1.24] |
| ρ_r | 0.68 | 0.67 | 0.68 | 0.04 | [0.60,0.74] |
| α_π | 1.49 | 1.57 | 1.56 | 0.16 | [1.27,1.88] |
| α_y | 0.15 | 0.15 | 0.14 | 0.05 | [0.05,0.24] |
| ρ_a | 0.99 | 0.98 | 0.99 | 0.01 | [0.96,1.00] |
| ρ_χ | 0.90 | 0.90 | 0.90 | 0.03 | [0.84,0.96] |
| σ_a | 0.21 | 0.24 | 0.23 | 0.07 | [0.12,0.39] |
| σ_χ | 0.17 | 0.19 | 0.18 | 0.03 | [0.12,0.26] |
| σ_r | 0.34 | 0.35 | 0.35 | 0.03 | [0.30,0.41] |
| σ_ν | 0.15 | 0.15 | 0.15 | 0.03 | [0.10,0.20] |
| η_y | 0.18 | 0.22 | 0.20 | 0.08 | [0.10,0.38] |
| Average log marginal density: | | | | | -324.209 |

3 Detailed Tables, Post-1982

Table 15: Posteriors – Perfect Info, Forward NK (Post 82 period)

| Param. | Mode | Mean | Median | Std. Dev. | 95% HPDI |
|-------------------------------|------|------|--------|-----------|-------------|
| ξ | 0.62 | 0.75 | 0.76 | 0.09 | [0.57,0.91] |
| φ | 0.27 | 0.24 | 0.23 | 0.12 | [0.02,0.46] |
| r^* | 0.15 | 0.75 | 0.73 | 0.38 | [0.08,1.46] |
| π^* | 0.01 | 0.52 | 0.50 | 0.30 | [0.00,1.04] |
| ρ_r | 0.57 | 0.37 | 0.38 | 0.14 | [0.10,0.61] |
| α_π | 2.15 | 2.25 | 2.25 | 0.15 | [1.98,2.55] |
| α_y | 0.13 | 0.13 | 0.13 | 0.05 | [0.04,0.23] |
| ρ_a | 0.99 | 0.99 | 0.99 | 0.01 | [0.97,1.00] |
| ρ_χ | 0.99 | 0.98 | 0.98 | 0.01 | [0.95,1.00] |
| σ_a | 0.37 | 1.12 | 1.07 | 0.54 | [0.28,2.22] |
| σ_χ | 0.11 | 0.12 | 0.12 | 0.01 | [0.10,0.15] |
| σ_r | 0.27 | 0.36 | 0.35 | 0.06 | [0.25,0.48] |
| σ_ν | 0.13 | 0.15 | 0.14 | 0.03 | [0.09,0.21] |
| Average log marginal density: | | | | | -111.093007 |

Table 16: Posteriors – Perfect Info, Hybrid NK (Post 82 period)

| Param. | Mode | Mean | Median | Std. Dev. | 95% HPDI |
|-------------------------------|------|------|--------|-----------|-------------|
| θ | 0.58 | 0.89 | 0.91 | 0.09 | [0.76,0.99] |
| ξ | 0.47 | 0.14 | 0.12 | 0.09 | [0.02,0.27] |
| φ | 0.26 | 0.28 | 0.28 | 0.11 | [0.06,0.51] |
| r^* | 0.93 | 0.80 | 0.80 | 0.22 | [0.34,1.20] |
| π^* | 0.72 | 0.76 | 0.76 | 0.18 | [0.39,1.14] |
| ρ_r | 0.70 | 0.80 | 0.80 | 0.05 | [0.69,0.90] |
| α_π | 1.94 | 1.52 | 1.50 | 0.23 | [1.08,1.99] |
| α_y | 0.14 | 0.09 | 0.09 | 0.05 | [0.01,0.18] |
| ρ_a | 0.98 | 0.14 | 0.08 | 0.20 | [0.00,0.74] |
| ρ_χ | 0.87 | 0.63 | 0.64 | 0.13 | [0.36,0.86] |
| σ_a | 0.15 | 0.10 | 0.10 | 0.03 | [0.07,0.14] |
| σ_χ | 0.11 | 0.27 | 0.26 | 0.08 | [0.12,0.41] |
| σ_r | 0.23 | 0.20 | 0.20 | 0.02 | [0.16,0.24] |
| σ_ν | 0.15 | 0.13 | 0.13 | 0.02 | [0.10,0.17] |
| Average log marginal density: | | | | | -111.131 |

Table 17: Posteriors – Imperfect Info, Pers. vs Temp. (Post 82 period)

| Param. | Mode | Mean | Median | Std. Dev. | 95% HPDI |
|-------------------------------|------|------|--------|-----------|-------------|
| ξ | 0.65 | 0.66 | 0.66 | 0.10 | [0.45,0.85] |
| φ | 0.26 | 0.25 | 0.24 | 0.11 | [0.03,0.46] |
| r^* | 0.15 | 0.78 | 0.77 | 0.37 | [0.09,1.45] |
| π^* | 0.00 | 0.55 | 0.54 | 0.30 | [0.01,1.08] |
| ρ_r | 0.51 | 0.47 | 0.48 | 0.11 | [0.24,0.65] |
| α_π | 2.16 | 2.20 | 2.19 | 0.15 | [1.90,2.48] |
| α_y | 0.13 | 0.14 | 0.14 | 0.05 | [0.05,0.24] |
| ρ_a | 0.99 | 0.99 | 0.99 | 0.01 | [0.97,1.00] |
| ρ_χ | 0.99 | 0.98 | 0.98 | 0.01 | [0.95,1.00] |
| σ_a | 0.45 | 0.61 | 0.52 | 0.37 | [0.15,1.30] |
| σ_χ | 0.11 | 0.12 | 0.12 | 0.01 | [0.10,0.15] |
| σ_r | 0.29 | 0.32 | 0.31 | 0.05 | [0.24,0.43] |
| σ_ν | 0.17 | 0.18 | 0.18 | 0.04 | [0.11,0.27] |
| Average log marginal density: | | | | | -108.155552 |

Table 18: Posteriors – Imperfect info, Inflation target shock (Post 82 period)

| Param. | Mode | Mean | Median | Std. Dev. | 95% HPDI |
|-------------------------------|------|------|--------|-----------|-------------|
| ξ | 0.63 | 0.67 | 0.68 | 0.09 | [0.50,0.83] |
| φ | 0.24 | 0.25 | 0.24 | 0.12 | [0.02,0.46] |
| r^* | 0.13 | 0.76 | 0.74 | 0.37 | [0.07,1.44] |
| π^* | 0.01 | 0.52 | 0.51 | 0.30 | [0.00,1.04] |
| ρ_r | 0.40 | 0.30 | 0.31 | 0.12 | [0.06,0.53] |
| α_π | 2.20 | 2.25 | 2.24 | 0.15 | [1.96,2.54] |
| α_y | 0.13 | 0.13 | 0.13 | 0.05 | [0.04,0.22] |
| ρ_a | 0.99 | 0.99 | 0.99 | 0.01 | [0.97,1.00] |
| ρ_χ | 0.99 | 0.98 | 0.98 | 0.01 | [0.95,1.00] |
| ρ_π | 0.36 | 0.31 | 0.31 | 0.11 | [0.10,0.55] |
| σ_a | 0.44 | 0.62 | 0.57 | 0.22 | [0.27,1.09] |
| σ_χ | 0.12 | 0.13 | 0.13 | 0.01 | [0.10,0.15] |
| σ_r | 0.21 | 0.21 | 0.21 | 0.06 | [0.11,0.32] |
| σ_π | 0.19 | 0.20 | 0.20 | 0.03 | [0.13,0.26] |
| Average log marginal density: | | | | | -109.042291 |

Table 19: Posteriors – Imperfect info, Measurement errors (Post 82 period)

| Param. | Mode | Mean | Median | Std. Dev. | 95% HPDI |
|-------------------------------|------|------|--------|-----------|---------------|
| ξ | 0.17 | 0.17 | 0.17 | 0.04 | [0.10, 0.26] |
| φ | 0.26 | 0.29 | 0.28 | 0.11 | [0.07, 0.51] |
| r^* | 0.97 | 0.95 | 0.96 | 0.19 | [0.60, 1.33] |
| π^* | 0.75 | 0.75 | 0.75 | 0.15 | [0.46, 1.04] |
| ρ_r | 0.38 | 0.38 | 0.39 | 0.09 | [0.20, 0.56] |
| α_π | 1.70 | 1.73 | 1.73 | 0.22 | [1.30, 2.17] |
| α_y | 0.18 | 0.19 | 0.19 | 0.04 | [0.10, 0.27] |
| ρ_a | 0.89 | 0.89 | 0.90 | 0.03 | [0.84, 0.94] |
| ρ_χ | 0.92 | 0.92 | 0.92 | 0.03 | [0.87, 0.96] |
| σ_a | 0.09 | 0.10 | 0.10 | 0.01 | [0.07, 0.13] |
| σ_χ | 0.20 | 0.22 | 0.21 | 0.03 | [0.16, 0.29] |
| σ_r | 0.12 | 0.12 | 0.12 | 0.02 | [0.09, 0.15] |
| σ_ν | 0.16 | 0.16 | 0.16 | 0.02 | [0.13, 0.20] |
| η_y | 0.20 | 0.27 | 0.24 | 0.13 | [0.11, 0.52] |
| η_π | 2.97 | 4.43 | 4.04 | 2.22 | [0.87, 8.78] |
| Average log marginal density: | | | | | -107.385365 |

Table 20: Moments (Post 82 period)

| | Data | (1) | (2) | (3) | (4) | (5) |
|----------------|-------|---------------|--------------|---------------|---------------|-------------|
| σ_y | 2.08 | 5.38 | 3.64 | 5.53 | 5.52 | 1.76 |
| | | [2.20,9.81] | [1.82,6.14] | [2.34,10.06] | [2.30,9.92] | [1.36,2.19] |
| σ_π | 0.28 | 0.62 | 0.64 | 0.60 | 0.64 | 0.37 |
| | | [0.32,1.10] | [0.42,0.97] | [0.31,1.05] | [0.32,1.12] | [0.29,0.46] |
| σ_R | 0.65 | 1.32 | 0.90 | 1.25 | 1.34 | 0.64 |
| | | [0.58,2.43] | [0.65,1.22] | [0.55,2.25] | [0.58,2.51] | [0.46,0.86] |
| $\rho(\pi, y)$ | -0.05 | -0.07 | 0.35 | -0.07 | -0.06 | 0.31 |
| | | [-0.13,-0.01] | [-0.21,0.72] | [-0.14,-0.01] | [-0.12,-0.01] | [0.07,0.55] |
| $\rho(R, y)$ | 0.16 | -0.09 | -0.01 | -0.10 | -0.09 | 0.42 |
| | | [-0.17,-0.02] | [-0.42,0.42] | [-0.18,-0.02] | [-0.17,-0.02] | [0.14,0.69] |
| $\rho_y(1)$ | 0.94 | 0.99 | 0.97 | 0.98 | 0.99 | 0.90 |
| | | [0.97,1.00] | [0.94,1.00] | [0.96,1.00] | [0.97,1.00] | [0.85,0.94] |
| $\rho_\pi(1)$ | 0.62 | 0.85 | 0.87 | 0.87 | 0.88 | 0.72 |
| | | [0.69,0.99] | [0.79,0.96] | [0.73,0.99] | [0.74,0.99] | [0.61,0.84] |
| $\rho_R(1)$ | 0.90 | 0.98 | 0.95 | 0.97 | 0.98 | 0.94 |
| | | [0.94,1.00] | [0.91,0.98] | [0.94,1.00] | [0.95,1.00] | [0.90,0.97] |
| $\rho_y(2)$ | 0.86 | 0.97 | 0.90 | 0.97 | 0.97 | 0.81 |
| | | [0.94,1.00] | [0.82,0.99] | [0.93,1.00] | [0.94,1.00] | [0.73,0.89] |
| $\rho_\pi(2)$ | 0.56 | 0.82 | 0.73 | 0.83 | 0.84 | 0.66 |
| | | [0.63,0.99] | [0.58,0.91] | [0.65,0.98] | [0.65,0.99] | [0.53,0.78] |
| $\rho_R(2)$ | 0.76 | 0.96 | 0.86 | 0.95 | 0.96 | 0.86 |
| | | [0.90,1.00] | [0.79,0.94] | [0.89,1.00] | [0.90,1.00] | [0.79,0.94] |
| $\rho_y(4)$ | 0.59 | 0.95 | 0.72 | 0.95 | 0.95 | 0.67 |
| | | [0.88,1.00] | [0.52,0.95] | [0.88,1.00] | [0.88,1.00] | [0.55,0.79] |
| $\rho_\pi(4)$ | 0.55 | 0.79 | 0.49 | 0.79 | 0.79 | 0.55 |
| | | [0.56,0.98] | [0.25,0.81] | [0.58,0.98] | [0.57,0.98] | [0.41,0.69] |
| $\rho_R(4)$ | 0.57 | 0.92 | 0.66 | 0.91 | 0.92 | 0.73 |
| | | [0.81,1.00] | [0.50,0.82] | [0.80,0.99] | [0.81,1.00] | [0.59,0.86] |

Note: (1): Baseline NK, (2): Hybrid NK (Backward Indexation, Real Rigidities), (3): Imperfect Info. Temporary vs Permanent Shocks, (4): Imperfect Info., Cogley–Sbordone, (5): Imperfect Info., Noisy Signals. 95% HPDI in brackets.

Table 21: HP-filtered Moments (Post 82 period)

| | Data | (1) | (2) | (3) | (4) | (5) |
|----------------|------|---------------|--------------|---------------|---------------|--------------|
| σ_y | 1.00 | 0.96 | 1.72 | 0.99 | 1.00 | 0.95 |
| | | [0.81,1.13] | [1.14,2.32] | [0.83,1.17] | [0.84,1.17] | [0.81,1.10] |
| σ_π | 0.18 | 0.23 | 0.42 | 0.22 | 0.23 | 0.24 |
| | | [0.20,0.27] | [0.33,0.53] | [0.19,0.26] | [0.20,0.27] | [0.20,0.27] |
| σ_R | 0.29 | 0.29 | 0.52 | 0.29 | 0.29 | 0.31 |
| | | [0.25,0.33] | [0.39,0.65] | [0.25,0.34] | [0.25,0.34] | [0.26,0.36] |
| $\rho(\pi, y)$ | 0.13 | 0.05 | 0.38 | 0.07 | 0.10 | 0.24 |
| | | [-0.03,0.15] | [0.02,0.65] | [-0.03,0.16] | [0.02,0.20] | [0.14,0.35] |
| $\rho(R, y)$ | 0.58 | -0.11 | -0.04 | -0.16 | -0.10 | 0.26 |
| | | [-0.21,-0.03] | [-0.38,0.27] | [-0.27,-0.05] | [-0.20,-0.02] | [0.05,0.47] |
| $\rho_y(1)$ | 0.84 | 0.71 | 0.90 | 0.68 | 0.70 | 0.67 |
| | | [0.69,0.72] | [0.85,0.94] | [0.64,0.72] | [0.69,0.72] | [0.64,0.70] |
| $\rho_\pi(1)$ | 0.13 | 0.21 | 0.73 | 0.29 | 0.34 | 0.34 |
| | | [0.11,0.32] | [0.67,0.81] | [0.17,0.41] | [0.23,0.44] | [0.23,0.45] |
| $\rho_R(1)$ | 0.84 | 0.67 | 0.86 | 0.67 | 0.72 | 0.75 |
| | | [0.61,0.72] | [0.81,0.91] | [0.61,0.73] | [0.67,0.77] | [0.69,0.81] |
| $\rho_y(2)$ | 0.65 | 0.47 | 0.70 | 0.44 | 0.47 | 0.42 |
| | | [0.45,0.48] | [0.61,0.79] | [0.40,0.48] | [0.45,0.48] | [0.37,0.46] |
| $\rho_\pi(2)$ | 0.08 | 0.08 | 0.47 | 0.12 | 0.13 | 0.20 |
| | | [0.03,0.14] | [0.37,0.61] | [0.05,0.19] | [0.06,0.20] | [0.14,0.28] |
| $\rho_R(2)$ | 0.55 | 0.44 | 0.66 | 0.44 | 0.48 | 0.50 |
| | | [0.39,0.48] | [0.58,0.75] | [0.39,0.49] | [0.44,0.52] | [0.44,0.57] |
| $\rho_y(4)$ | 0.18 | 0.12 | 0.25 | 0.11 | 0.12 | 0.06 |
| | | [0.11,0.12] | [0.11,0.40] | [0.09,0.12] | [0.11,0.12] | [0.02,0.10] |
| $\rho_\pi(4)$ | 0.27 | -0.02 | 0.06 | -0.02 | -0.03 | 0.01 |
| | | [-0.04,0.00] | [-0.12,0.19] | [-0.04,0.01] | [-0.06,-0.01] | [-0.01,0.04] |
| $\rho_R(4)$ | 0.12 | 0.11 | 0.24 | 0.11 | 0.12 | 0.11 |
| | | [0.09,0.12] | [0.09,0.38] | [0.08,0.13] | [0.10,0.14] | [0.06,0.15] |

Note: (1): Baseline NK, (2): Hybrid NK (Backward Indexation, Real Rigidities), (3): Imperfect Info. Temporary vs Permanent Shocks, (4): Imperfect Info., Cogley–Sbordone, (5): Imperfect Info., Noisy Signals. 95% HPDI in brackets.

3.1 Figures, Post 1982 period

Figure 1: Impulse Response Functions – Perfect info, forward NK (Post 82 period)

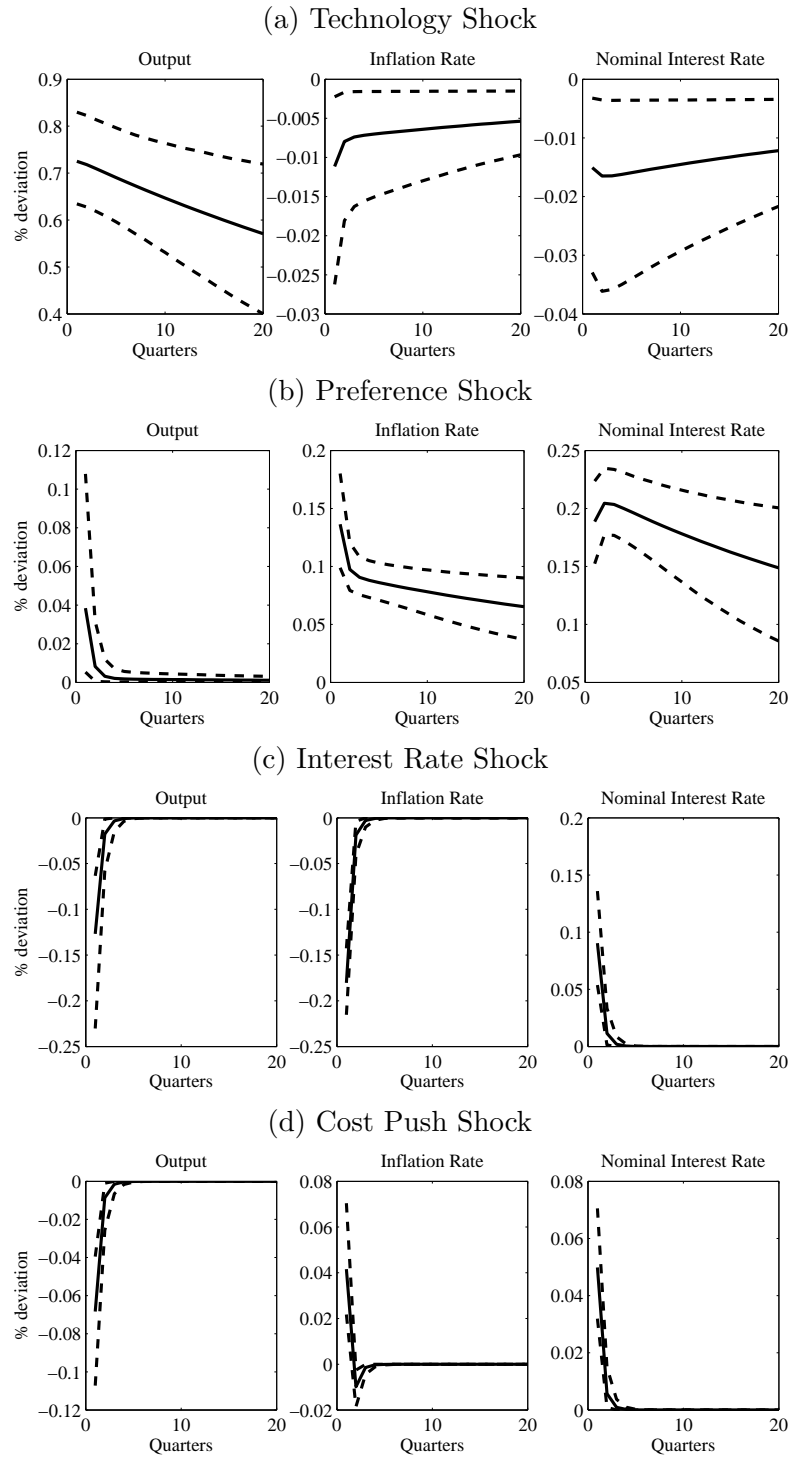


Figure 2: Impulse Response Functions – Perfect info, hybrid NK (Post 82 period)

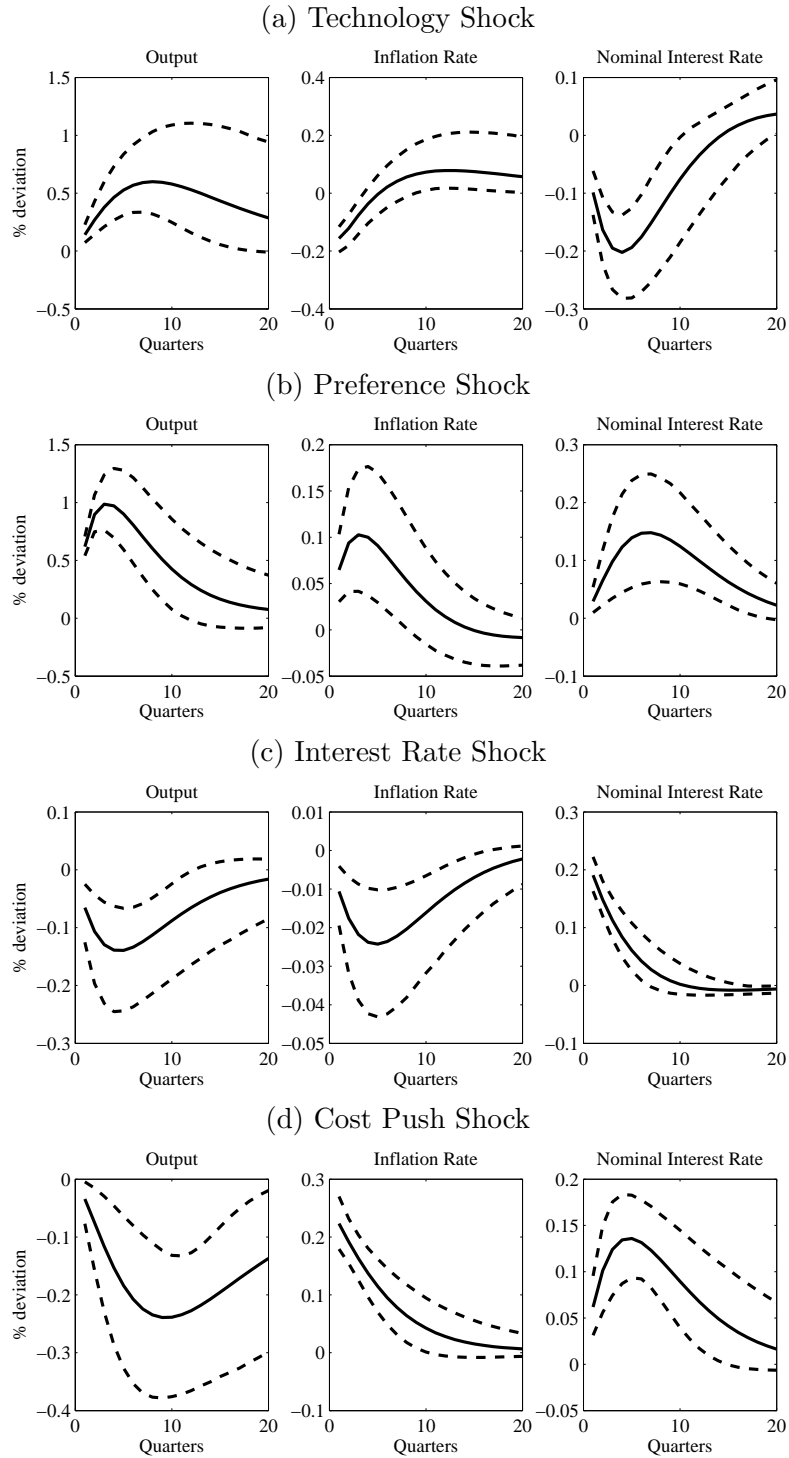


Figure 3: Impulse Response Functions – Imperfect info, Pers. vs Temp. (Post 82 period)

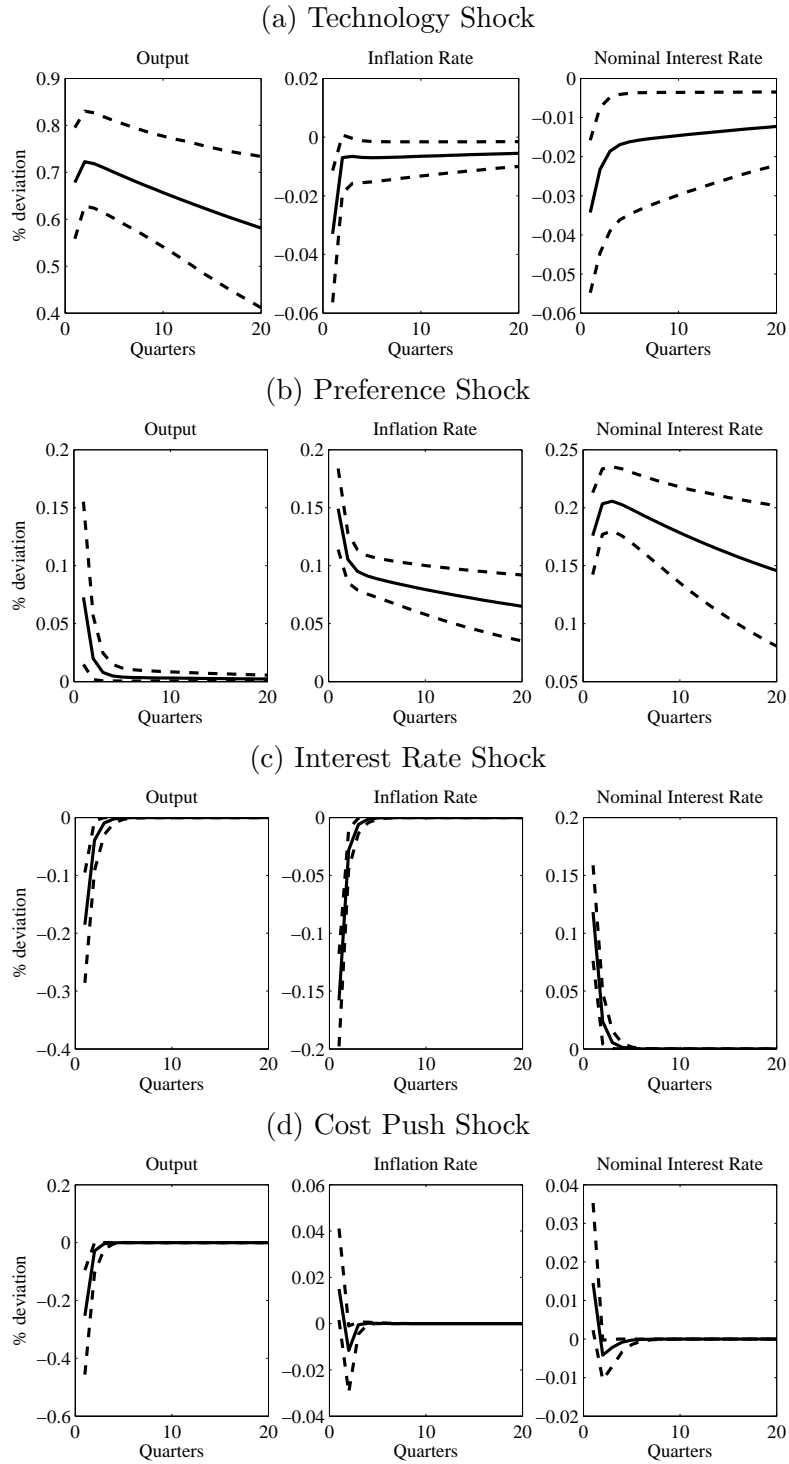


Figure 4: Impulse Response Functions– Imperfect info, Inflation target shock (Post 82 period)

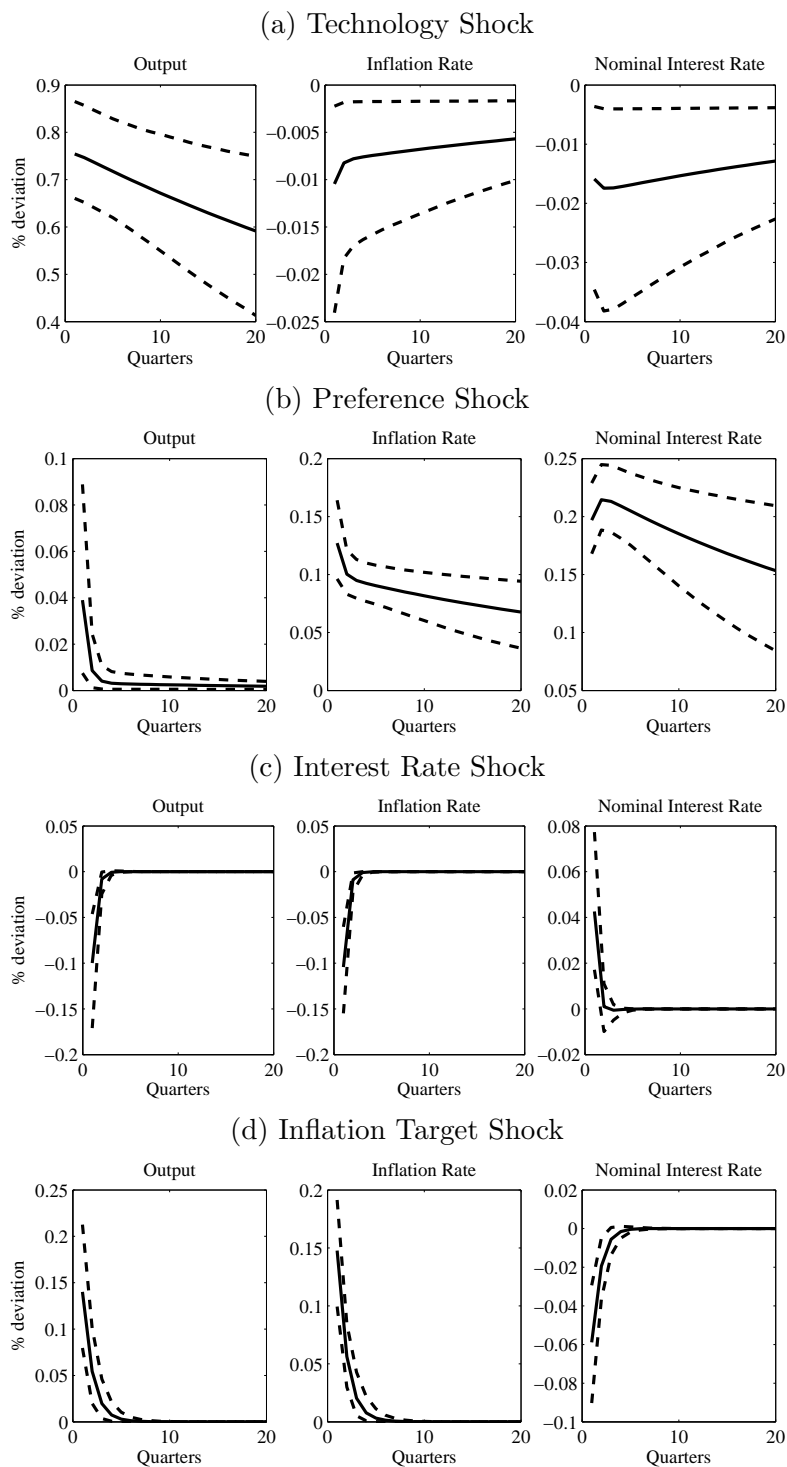
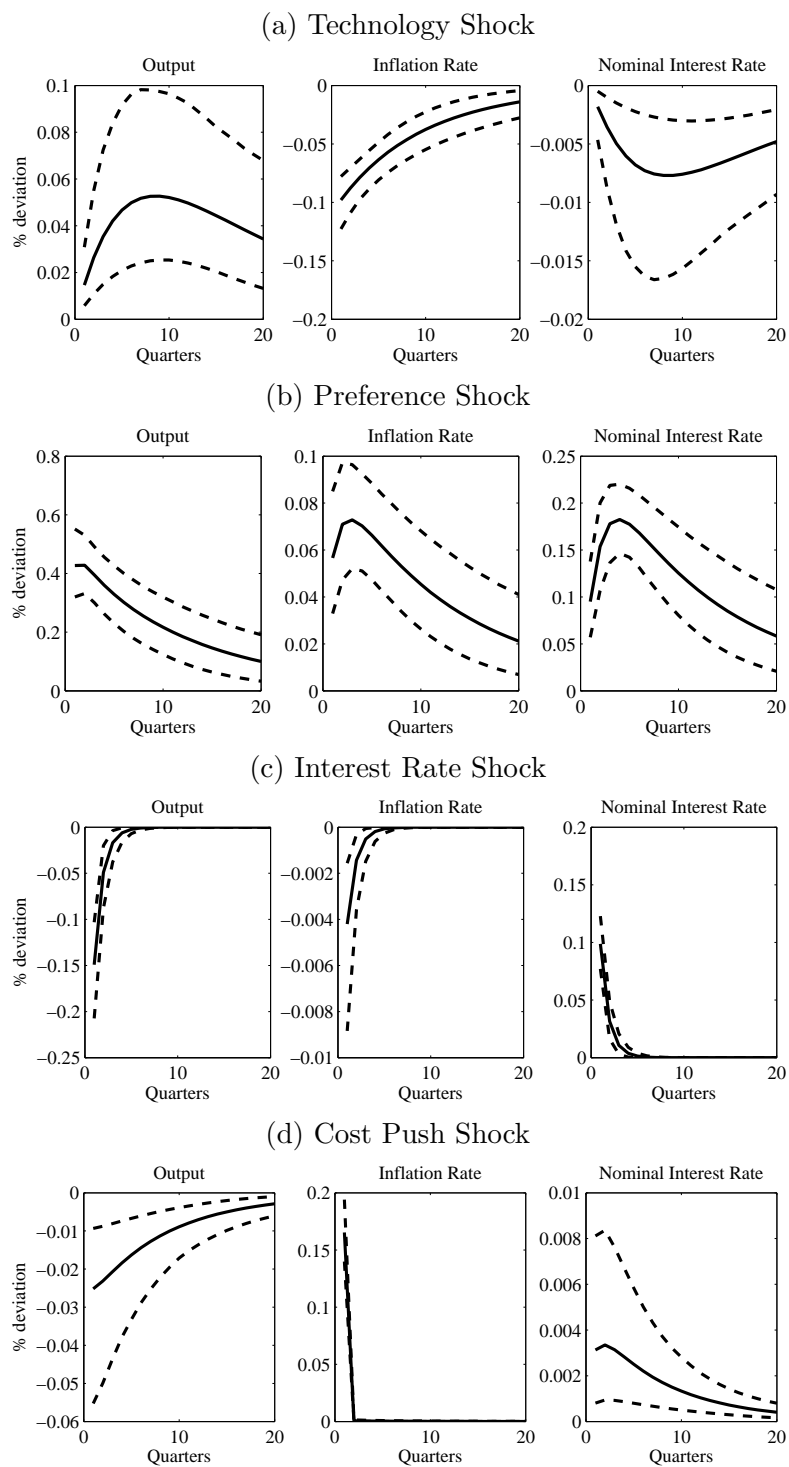


Figure 5: Impulse Response Functions – Imperfect info, measurement errors (Post 82 period)



4 Solution Method

Let the state of the economy be represented by two vectors \tilde{X}_t^b and \tilde{X}_t^f . The first one includes the predetermined (backward looking) state variables, i.e. $\tilde{X}_t^b = (\tilde{R}_{t-1}, \tilde{z}_t, \tilde{g}_t, \tilde{\varepsilon}_t^R)'$, whereas the second one consists of the forward looking state variables, i.e. $\tilde{X}_t^f = (\tilde{y}_t, \tilde{\pi}_t)'$. The model admits the following representation

$$M_0 \begin{pmatrix} \tilde{X}_{t+1}^b \\ \mathbb{E}_t \tilde{X}_{t+1}^f \end{pmatrix} + M_1 \begin{pmatrix} \tilde{X}_t^b \\ \tilde{X}_t^f \end{pmatrix} = M_2 \varepsilon_{t+1} \quad (1)$$

Let us denote the signal process by $\{S_t\}$. The measurement equation relates the state of the economy to the signal:

$$S_t = C \begin{pmatrix} \tilde{X}_t^b \\ \tilde{X}_t^f \end{pmatrix} + v_t. \quad (2)$$

Finally u and v are assumed to be normally distributed covariance matrices Σ_{uu} and Σ_{vv} respectively and $E(uv') = 0$.

$X_{t+i|t} = E(X_{t+i} | \mathcal{I}_t)$ for $i \geq 0$ and where \mathcal{I}_t denotes the information set available to the agents at the beginning of period t . The information set available to the agents consists of *i*) the structure of the model and *ii*) the history of the observable signals they are given in each period:

$$\mathcal{I}_t = \{S_{t-j}, j \geq 0, M_0, M_1, M_2, C, \Sigma_{uu}, \Sigma_{vv}\}$$

The information structure of the agents is described fully by the specification of the signals.

4.1 Solving the system

Step 1: We first solve for the expected system:

$$M_0 \begin{pmatrix} X_{t+1|t}^b \\ X_{t+1|t}^f \end{pmatrix} + M_1 \begin{pmatrix} X_{t|t}^b \\ X_{t|t}^f \end{pmatrix} = \quad (3)$$

which rewrites as

$$\begin{pmatrix} X_{t+1|t}^b \\ X_{t+1|t}^f \end{pmatrix} = W \begin{pmatrix} X_{t|t}^b \\ X_{t|t}^f \end{pmatrix} \quad (4)$$

where

$$W = -M_0^{-1} M_1$$

After getting the Jordan form associated to (4) and applying standard methods for eliminating bubbles, we get

$$X_{t|t}^f = G X_{t|t}^b$$

From which we get

$$X_{t+1|t}^b = (W_{bb} + W_{bf}G) X_{t|t}^b = W^b X_{t|t}^b \quad (5)$$

$$X_{t+1|t}^f = (W_{fb} + W_{ff}G) X_{t|t}^b = W^f X_{t|t}^b \quad (6)$$

Step 2: We have

$$M_0 \begin{pmatrix} X_{t+1}^b \\ X_{t+1|t}^f \end{pmatrix} + M_1 \begin{pmatrix} X_t^b \\ X_t^f \end{pmatrix} = M_2 u_{t+1}$$

Taking expectations, we have

$$M_0 \begin{pmatrix} X_{t+1|t}^b \\ X_{t+1|t}^f \end{pmatrix} + M_1 \begin{pmatrix} X_{t|t}^b \\ X_{t|t}^f \end{pmatrix} = 0$$

Subtracting, we get

$$M_0 \begin{pmatrix} X_{t+1}^b - X_{t+1|t}^b \\ 0 \end{pmatrix} + M_1 \begin{pmatrix} X_t^b - X_{t|t}^b \\ X_t^f - X_{t|t}^f \end{pmatrix} = M_2 u_{t+1} \quad (7)$$

which rewrites

$$\begin{pmatrix} X_{t+1}^b - X_{t+1|t}^b \\ 0 \end{pmatrix} = W^c \begin{pmatrix} X_t^b - X_{t|t}^b \\ X_t^f - X_{t|t}^f \end{pmatrix} + M_0^{-1} M_2 u_{t+1} \quad (8)$$

where, $W^c = -M_0^{-1} M_1$. Hence, considering the second block of the above matrix equation, we get

$$W_{fb}^c (X_t^b - X_{t|t}^b) + W_{ff}^c (X_t^f - X_{t|t}^f) = 0$$

which gives

$$X_t^f = F^0 X_t^b + F^1 X_{t|t}^b$$

with $F^0 = -W_{ff}^c^{-1} W_{fb}^c$ and $F^1 = G - F^0$.

Now considering the first block, we have

$$X_{t+1}^b = X_{t+1|t}^b + W_{bb}^c (X_t^b - X_{t|t}^b) + W_{bf}^c (X_t^f - X_{t|t}^f) + M^2 u_{t+1}$$

from which we get, using (5)

$$X_{t+1}^b = M^0 X_t^b + M^1 X_{t|t}^b + M^2 u_{t+1}$$

with $M^0 = W_{bb}^c + W_{bf}^c F^0$, $M^1 = W^b - M^0$ and $M^2 = M_0^{-1} M_2$.

We also have

$$S_t = C_b X_t^b + C_f X_t^f + v_t$$

from which we get

$$S_t = S^0 X_t^b + S^1 X_{t|t}^b + v_t$$

where $S^0 = C_b + C_f F^0$ and $S^1 = C_f F^1$

4.2 Filtering

Since our solution involves terms in $X_{t|t}^b$, we would like to compute this quantity. However, the only information we can exploit is a signal S_t that was described previously. We therefore use a Kalman filter approach to compute the optimal prediction of $X_{t|t}^b$.

In order to recover the Kalman filter, it is a good idea to think in terms of expectation errors. Therefore, let us define

$$\tilde{X}_t^b = X_t^b - X_{t|t-1}^b$$

and

$$\tilde{S}_t = S_t - S_{t|t-1}$$

Note that since S_t depends on $X_{t|t}^b$, only the signal relying on $\tilde{S}_t = S_t - S^1 X_{t|t}^b$ can be used to infer anything on $X_{t|t}^b$. Therefore, the policy maker revises its expectations using a linear rule depending on $\tilde{S}_t^e = S_t - S^1 X_{t|t}^b$. The filtering equation then writes

$$X_{t|t}^b = X_{t|t-1}^b + K(\tilde{S}_t^e - \tilde{S}_{t|t-1}^e) = X_{t|t-1}^b + K(S^0 \tilde{X}_t^b + v_t)$$

where K is the filter gain matrix, that we would like to compute.

The first thing we have to do is to rewrite the system in terms of state-space representation. Since $S_{t|t-1} = (S^0 + S^1)X_{t|t-1}^b$, we have

$$\begin{aligned} \tilde{S}_t &= S^0(X_t^b - X_{t|t}^b) + S^1(X_{t|t}^b - X_{t|t-1}^b) + v_t \\ &= S^0 \tilde{X}_t^b + S^1 K(S^0 \tilde{X}_t^b + v_t) + v_t \\ &= S^* \tilde{X}_t^b + \nu_t \end{aligned}$$

where $S^* = (I + S^1 K)S^0$ and $\nu_t = (I + S^1 K)v_t$.

Now, consider the law of motion of backward state variables, we get

$$\begin{aligned} \tilde{X}_{t+1}^b &= M^0(X_t^b - X_{t|t}^b) + M^2 u_{t+1} \\ &= M^0(X_t^b - X_{t|t-1}^b - X_{t|t}^b + X_{t|t-1}^b) + M^2 u_{t+1} \\ &= M^0 \tilde{X}_t^b - M^0(X_{t|t}^b + X_{t|t-1}^b) + M^2 u_{t+1} \\ &= M^0 \tilde{X}_t^b - M^0 K(S^0 \tilde{X}_t^b + v_t) + M^2 u_{t+1} \\ &= M^* \tilde{X}_t^b + \omega_{t+1} \end{aligned}$$

where $M^* = M^0(I - K S^0)$ and $\omega_{t+1} = M^2 u_{t+1} - M^0 K v_t$.

We therefore end-up with the following state-space representation

$$\tilde{X}_{t+1}^b = M^* \tilde{X}_t^b + \omega_{t+1} \tag{9}$$

$$\tilde{S}_t = S^* \tilde{X}_t^b + \nu_t \tag{10}$$

For which the Kalman filter is given by

$$\tilde{X}_{t|t}^b = \tilde{X}_{t|t-1}^b + PS^{*'}(S^*PS^{*'} + \Sigma_{\nu\nu})^{-1}(S^*\tilde{X}_t^b + \nu_t)$$

But since $\tilde{X}_{t|t}^b$ is an expectation error, it is not correlated with the information set in $t-1$, such that $\tilde{X}_{t|t-1}^b = 0$. The prediction formula for $\tilde{X}_{t|t}^b$ therefore reduces to

$$\tilde{X}_{t|t}^b = PS^{*'}(S^*PS^{*'} + \Sigma_{\nu\nu})^{-1}(S^*\tilde{X}_t^b + \nu_t) \quad (11)$$

where P solves

$$P = M^*PM^{*'} + \Sigma_{\omega\omega}$$

and $\Sigma_{\nu\nu} = (I + S^1K)\Sigma_{vv}(I + S^1K)'$ and $\Sigma_{\omega\omega} = M^0K\Sigma_{vv}K'M^{0'} + M^2\Sigma_{uu}M^{2'}$

Note however that the above solution is obtained for a given K matrix that remains to be computed. We can do that by using the basic equation of the Kalman filter:

$$\begin{aligned} X_{t|t}^b &= X_{t|t-1}^b + K(\tilde{S}_t^e - \tilde{S}_{t|t-1}^e) \\ &= X_{t|t-1}^b + K(S_t - S^1X_{t|t}^b - (S_{t|t-1} - S^1X_{t|t-1}^b)) \\ &= X_{t|t-1}^b + K(S_t - S^1X_{t|t}^b - S^0X_{t|t-1}^b) \end{aligned}$$

Solving for $X_{t|t}^b$, we get

$$\begin{aligned} X_{t|t}^b &= (I + KS^1)^{-1}(X_{t|t-1}^b + K(S_t - S^0X_{t|t-1}^b)) \\ &= (I + KS^1)^{-1}(X_{t|t-1}^b + KS^1X_{t|t-1}^b - KS^1X_{t|t-1}^b + K(S_t - S^0X_{t|t-1}^b)) \\ &= (I + KS^1)^{-1}(I + KS^1)X_{t|t-1}^b + (I + KS^1)^{-1}K(S_t - (S^0 + S^1)X_{t|t-1}^b) \\ &= X_{t|t-1}^b + (I + KS^1)^{-1}K\tilde{S}_t \\ &= X_{t|t-1}^b + K(I + S^1K)^{-1}\tilde{S}_t \\ &= X_{t|t-1}^b + K(I + S^1K)^{-1}(S^*\tilde{X}_t^b + \nu_t) \end{aligned}$$

where we made use of the identity $(I + KS^1)^{-1}K \equiv K(I + S^1K)^{-1}$. Hence, identifying to (11), we have

$$K(I + S^1K)^{-1} = PS^{*'}(S^*PS^{*'} + \Sigma_{\nu\nu})^{-1}$$

remembering that $S^* = (I + S^1K)S^0$ and $\Sigma_{\nu\nu} = (I + S^1K)\Sigma_{vv}(I + S^1K)'$, we have

$$K(I + S^1K)^{-1} = PS^{0'}(I + S^1K)'((I + S^1K)S^0PS^{0'}(I + S^1K)' + (I + S^1K)\Sigma_{vv}(I + S^1K)')^{-1}(I + S^1K)S^0$$

which rewrites as

$$\begin{aligned} K(I + S^1K)^{-1} &= PS^{0'}(I + S^1K)' \left[(I + S^1K)(S^0PS^{0'} + \Sigma_{vv})(I + S^1K)' \right]^{-1} \\ K(I + S^1K)^{-1} &= PS^{0'}(I + S^1K)'(I + S^1K)'^{-1}(S^0PS^{0'} + \Sigma_{vv})^{-1}(I + S^1K)^{-1} \end{aligned}$$

Hence, we obtain

$$K = PS^{0'}(S^0PS^{0'} + \Sigma_{vv})^{-1} \quad (12)$$

Now, recall that

$$P = M^*PM^{*'} + \Sigma_{\omega\omega}$$

Remembering that $M^* = M^0(I + KS^0)$ and $\Sigma_{\omega\omega} = M^0K\Sigma_{vv}K'M^{0'} + M^2\Sigma_{uu}M^{2'}$, we have

$$\begin{aligned} P &= M^0(I - KS^0)P[M^0(I - KS^0)]' + M^0K\Sigma_{vv}K'M^{0'} + M^2\Sigma_{uu}M^{2'} \\ &= M^0 \left[(I - KS^0)P(I - S^{0'}K') + K\Sigma_{vv}K' \right] M^{0'} + M^2\Sigma_{uu}M^{2'} \end{aligned}$$

Plugging the definition of K in the latter equation, we obtain

$$P = M^0 \left[P - PS^{0'}(S^0PS^{0'} + \Sigma_{vv})^{-1}S^0P \right] M^{0'} + M^2\Sigma_{uu}M^{2'} \quad (13)$$

4.3 Summary

We end-up with the system of equations:

$$X_{t+1}^b = M^0X_t^b + M^1X_{t|t}^b + M^2u_{t+1} \quad (14)$$

$$S_t = S_b^0X_t^b + S_b^1X_{t|t}^b + v_t \quad (15)$$

$$X_t^f = F^0X_t^b + F^1X_{t|t}^b \quad (16)$$

$$X_{t|t}^b = X_{t|t-1}^b + K(S^0(X_t^b - X_{t|t-1}^b) + v_t) \quad (17)$$

$$X_{t+1|t}^b = (M^0 + M^1)X_{t|t}^b \quad (18)$$

which fully describe the dynamics of our economy.

This may be recast as a standard state-space problem

$$\begin{aligned} X_{t+1|t+1}^b &= X_{t+1|t}^b + K(S^0(X_{t+1}^b - X_{t+1|t}^b) + v_{t+1}) \\ &= (M^0 + M^1)X_{t|t}^b + K(S^0(M^0X_t^b + M^1X_{t|t}^b + M^2u_{t+1} - (M^0 + M^1)X_{t|t}^b) + v_{t+1}) \\ &= KS^0M^0X_t^b + ((I - KS^0)M^0 + M^1)X_{t|t}^b + KS^0M^2u_{t+1} + Kv_{t+1} \end{aligned}$$

Then

$$\begin{pmatrix} X_{t+1}^b \\ X_{t+1|t+1}^b \end{pmatrix} = M_X \begin{pmatrix} X_t^b \\ X_{t|t}^b \end{pmatrix} + M_E \begin{pmatrix} u_{t+1} \\ v_{t+1} \end{pmatrix}$$

where

$$M_X = \begin{pmatrix} M^0 & M^1 \\ KS^0M^0 & ((I - KS^0)M^0 + M^1) \end{pmatrix} \text{ and } M_E = \begin{pmatrix} M^2 & 0 \\ KS^0M^2 & K \end{pmatrix}$$

and

$$X_t^f = M_F \begin{pmatrix} X_t^b \\ X_{t|t}^b \end{pmatrix}$$

where

$$M_F = \begin{pmatrix} F^0 & F^1 \end{pmatrix}$$