Technical Appendix:
Understanding Recent Economic History: Insights from Cross Country Comparisons

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A Solving the model

A.1 The Economy in the Pre–Technological Revolution

The households problem writes

$$\max \int_0^\infty e^{-\rho t} u(c(t)) L(t) \mathrm{d}t$$

s.t.

$$\begin{cases} \dot{B}(t) + c(t)L(t) + I^{\mathrm{O}}(t) = w^{\mathrm{O}}(t)L(t) + rB(t) + q^{\mathrm{O}}(t)K(t) \\ \dot{K}^{\mathrm{O}}(t) = I^{\mathrm{O}}(t) - \delta K^{\mathrm{O}}(t) \\ I^{\mathrm{O}}(t) \geqslant 0 \end{cases}$$

since we have $\dot{L}(t) = nL(t)$, the problem, in intensive form rewrites

$$\max \int_0^\infty e^{-(\rho-n)t} u(c(t)) \mathrm{d}t$$

s.t.

$$\begin{cases} \dot{b}(t) + c(t) + i^{\mathrm{O}}(t) = w^{\mathrm{O}}(t) + (r-n)b(t) + q^{\mathrm{O}}(t)k(t) \\ \dot{k}^{\mathrm{O}}(t) = i^{\mathrm{O}}(t) - (\delta+n)k^{\mathrm{O}}(t) \\ i^{\mathrm{O}}(t) \geqslant 0 \end{cases}$$

Forming the Hamiltonian of the system, and maximizing, we get the following set of first order conditions

$$u'(c(t)) = \lambda^b(t) \tag{A.1}$$

$$\lambda^b(t) = \lambda^k(t) + \lambda^i(t) \tag{A.2}$$

$$\dot{\lambda}^b(t) = (\rho - r)\lambda^b(t) \tag{A.3}$$

$$\dot{\lambda}^k(t) = (\rho+\delta)\lambda^k(t) - q^{\mathrm{O}}(t)\lambda^b(t) \tag{A.4}$$

$$\lambda^i(t) i^{\mathrm{O}}(t) = 0 \tag{A.5}$$

where $\lambda^b(t)$, $\lambda^k(t)$ and $\lambda^i(t)$ respectively denote the shadow price of bonds, capital and the lagrange multiplier associated with the irreversibility constraint. Finally, we have the following transversality conditions

$$\lim_{t\to\infty} e^{-(r-n)t}\lambda^b(t)b(t) = 0 \text{ and } \lim_{t\to\infty} e^{-(r-n)t}\lambda^k(t)k^{\mathrm{O}}(t) = 0$$

The firm maximizes profits, given by

$$K^{\mathrm{O}}(t)^{\alpha}L(t)^{1-\alpha} - q^{\mathrm{O}}(t)K^{\mathrm{O}}(t) - w^{\mathrm{O}}(t)L(t)$$

or, in intensive terms,

$$k^{\mathrm{O}}(t)^{\alpha} - q^{\mathrm{O}}(t)k^{\mathrm{O}}(t) - w^{\mathrm{O}}(t)$$

from which we get

$$q^{\mathrm{O}}(t) = \alpha k^{\mathrm{O}}(t)^{\alpha-1} \tag{A.6}$$

$$w^{\mathrm{O}}(t) = (1-\alpha)k^{\mathrm{O}}(t)^{\alpha} \tag{A.7}$$

From (A.3), we see that either $\rho > r$ and $\lambda^b(t)$ diverges, or $\rho < r$ and it converges toward zero, or $\rho = r$ and the economy jumps to its steady state level. The two first cases eventually violate the transversality condition, such that only the third solution is economically meaningful. Then, we have

$$q^{\mathrm{O}} = \alpha k^{\mathrm{O}\alpha-1} \tag{A.8}$$

$$w^{\mathrm{O}} = (1-\alpha)k^{\mathrm{O}\alpha} \tag{A.9}$$

$$u'(c) = \lambda^b \tag{A.10}$$

$$\lambda^b = \lambda^k + \lambda^i \tag{A.11}$$

$$(r+\delta)\lambda^k(t) = q^{\mathrm{O}}\lambda^b \tag{A.12}$$

$$c + i^{\mathrm{O}} = w^{\mathrm{O}} + (r-n)b + q^{\mathrm{O}}k^{\mathrm{O}} \tag{A.13}$$

$$i^{\mathrm{O}} = (\delta+n)k^{\mathrm{O}} \tag{A.14}$$

$$y^{\mathrm{O}} = k^{\mathrm{O}\alpha} \tag{A.15}$$

$$\lambda^i i^{\mathrm{O}} = 0 \tag{A.16}$$

From (A.8), it is optimal for the economy to have $k^{\mathrm{O}} > 0$ in the steady state, otherwise since $q^{\mathrm{O}} \longrightarrow \infty$ in that case, it would be optimal for the economy to build capital at an infinite pace. Hence, from (A.14), we see that $i^{\mathrm{O}} > 0$ so that $\lambda^{\mathrm{I}} = 0$. Therefore, from (A.11) and (A.12), we have

$$q^{\mathrm{O}} = r + \delta$$

such that from (A.8)

$$k^{\mathrm{O}} = \left(\frac{\alpha}{r+\delta}\right)^{\frac{1}{1-\alpha}}$$

Then, it is straightforward to obtain

$$y^{\mathrm{O}} = \left(\frac{\alpha}{r+\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

From this equation, we see that

$$\frac{\dot{y}^{\mathrm{O}}}{y^{\mathrm{O}}} = 0$$

and is therefore independent from the rate of population growth.

A.2 The Economy during the Technological Revolution

The households problem writes

$$\max \int_0^{\infty} e^{-\rho t} u(c(t)) L(t) \mathrm{d}t$$

s.t.

$$\dot{B}(t) + c(t)L(t) + I^{\mathrm{N}}(t) = r \quad B(t) + w^{\mathrm{O}}(t)(L(t) - S(t) - H(t)) + w^{\mathrm{N}}(t)S(t) + q^{\mathrm{O}}(t)K^{\mathrm{O}}(t) + q^{\mathrm{N}}(t)K^{\mathrm{N}}(t)$$

$$\dot{K}^{\mathrm{O}}(t) = I^{\mathrm{O}}(t) - \delta K^{\mathrm{O}}(t)$$

$$\dot{K}^{\mathrm{N}}(t) = I^{\mathrm{N}}(t) - \delta K^{\mathrm{N}}(t)$$

$$\dot{S}(t) = \Omega H(t) + n\overline{s}(t)L(t)$$

$$I^{\mathrm{O}}(t) \geqslant 0$$

$$I^{\mathrm{N}}(t) \geqslant 0$$

which rewrites, in intensive terms, as

$$\max \int_0^{\infty} e^{-(\rho-n)t} u(c(t)) \mathrm{d}t$$

s.t.

$$\begin{aligned} \dot{b}(t) + c(t) + i^{\mathrm{N}}(t) + i^{\mathrm{O}}(t) =& rb(t) + q^{\mathrm{O}}(t)k^{\mathrm{O}}(t) + q^{\mathrm{N}}(t)k^{\mathrm{N}}(t) \\ &+ w^{\mathrm{O}}(t)(1 - s(t) - h(t)) + w^{\mathrm{N}}(t)s(t) \end{aligned} \quad \text{(A.17)}$$

$$\dot{k}^{\mathrm{O}}(t) = i^{\mathrm{O}}(t) - (\delta + n)k^{\mathrm{O}}(t) \quad \text{(A.18)}$$

$$\dot{k}^{\mathrm{N}}(t) = i^{\mathrm{N}}(t) - (\delta + n)k^{\mathrm{N}}(t) \quad \text{(A.19)}$$

$$\dot{s}(t) = \Omega h(t) - ns(t) + n\overline{s}(t) \quad \text{(A.20)}$$

$$i^{\mathrm{O}}(t) \geqslant 0 \quad \text{(A.21)}$$

$$i^{\mathrm{N}}(t) \geqslant 0 \quad \text{(A.22)}$$

One can then form the following Hamiltonian

$$\begin{aligned}\mathscr{H} =& u(c(t)) + \lambda^b(t)\Big[rb(t) + w^{\mathrm{O}}(t)(1-s(t)-h(t)) + w^{\mathrm{N}}(t)s(t) + \\ & q^{\mathrm{O}}(t)k^{\mathrm{O}}(t) + q^{\mathrm{N}}(t)k^{\mathrm{N}}(t) - c(t) - i^{\mathrm{N}}(t) - i^{\mathrm{O}}(t)\Big] \\ & + \lambda^{\mathrm{KO}}(t)(i^{\mathrm{O}}(t) - (\delta+n)k^{\mathrm{O}}(t)) + \lambda^{\mathrm{KN}}(t)(i^{\mathrm{N}}(t) - (\delta+n)k^{\mathrm{N}}(t)) \\ & + \lambda^s(t)(\Omega h(t) - ns(t) + n\overline{s}(t)) + \lambda_i^{\mathrm{O}}(t)i^{\mathrm{O}}(t) + \lambda_i^{\mathrm{N}}(t)i^{\mathrm{N}}(t)\end{aligned}$$

from which it is clear that the labor market allocation problem reduces to the reduced Hamiltonian

$$\widetilde{\mathscr{H}} = \lambda^b(t)(w^{\mathrm{O}}(t)(1-s(t)-h(t)) + w^{\mathrm{N}}(t)s(t)) + \lambda^s(t)(\Omega h(t) - ns(t) + n\overline{s}(t))$$

or, renormalizing by $\lambda^b(t)$

$$\widehat{\mathscr{H}} = w^{\mathrm{O}}(t)(1-s(t)-h(t)) + w^{\mathrm{N}}(t)s(t) + \lambda^s(t)(\Omega h(t) - ns(t))$$

which corresponds to the labor allocation problem reported in the body text.

The optimality conditions associated to the main problem are

$$u'(c(t)) = \lambda^b(t) \tag{A.23}$$

$$\lambda^b(t) = \lambda^{\mathrm{KO}}(t) + \lambda_i^{\mathrm{O}}(t) \tag{A.24}$$

$$\lambda^b(t) = \lambda^{\mathrm{KN}}(t) + \lambda_i^{\mathrm{N}}(t) \tag{A.25}$$

$$\dot{\lambda}^b(t) = (\rho - r)\lambda^b(t) \tag{A.26}$$

$$\dot{\lambda}^{\mathrm{KO}}(t) = (\rho+\delta)\lambda^{\mathrm{KO}}(t) - q^{\mathrm{O}}(t)\lambda^b(t) \tag{A.27}$$

$$\dot{\lambda}^{\mathrm{KN}}(t) = (\rho+\delta)\lambda^{\mathrm{KN}}(t) - q^{\mathrm{N}}(t)\lambda^b(t) \tag{A.28}$$

$$\lambda^b(t)w^{\mathrm{O}}(t) = \Omega\lambda^s(t) \tag{A.29}$$

$$\dot{\lambda}^s(t) = -\lambda^b(t)(w^{\mathrm{N}}(t) - w^{\mathrm{O}}(t)) + r\lambda^s(t) \tag{A.30}$$

$$\lambda_i^{\mathrm{O}}(t)i^{\mathrm{O}}(t) = 0 \tag{A.31}$$

$$\lambda_i^{\mathrm{N}}(t)i^{\mathrm{N}}(t) = 0 \tag{A.32}$$

where $\lambda^b(t)$, $\lambda^{\mathrm{KO}}(t)$, $\lambda^{\mathrm{KN}}(t)$, $\lambda_i^{\mathrm{O}}(t)$ and $\lambda_i^{\mathrm{N}}(t)$ respectively denote the shadow price of bonds, capital in the old and the new technology and the lagrange multiplier associated with the irreversibility constraints on the old and new form of capital. Finally, we have the following transversality conditions.

$$\lim_{t\to\infty} e^{-(r-n)t}\lambda^b(t)b(t) = 0, \quad \lim_{t\to\infty} e^{-(r-n)t}\lambda^{\mathrm{KO}}(t)k^{\mathrm{O}}(t) = 0 \text{ and } \lim_{t\to\infty} e^{-(r-n)t}\lambda^{\mathrm{KN}}(t)k^{\mathrm{N}}(t) = 0$$

The firm decides on its production plan maximizing profits, which are given by — in intensive form

$$y^{\mathrm{O}}(t) + y^{\mathrm{N}}(t) - q^{\mathrm{O}}(t)k^{\mathrm{O}}(t) - w^{\mathrm{O}}(t)\ell^{\mathrm{O}}(t) - q^{\mathrm{N}}(t)k^{\mathrm{N}}(t) - w^{\mathrm{N}}(t)\ell^{\mathrm{N}}(t)$$

This leads to the following set of optimality conditions, expressed in intensive forms

$$q^{\mathrm{O}}(t) = \alpha k^{\mathrm{O}}(t)^{\alpha-1} \ell^{\mathrm{O}}(t)^{1-\alpha} \tag{A.33}$$

$$w^{\mathrm{O}}(t) = (1-\alpha) k^{\mathrm{O}}(t)^{\alpha} \ell^{\mathrm{O}}(t)^{-\alpha} \tag{A.34}$$

$$q^{\mathrm{N}}(t) = \alpha k^{\mathrm{N}}(t)^{\alpha-1} ((1+\gamma)\ell^{\mathrm{N}}(t))^{1-\alpha} \tag{A.35}$$

$$w^{\mathrm{N}}(t) = (1-\alpha) k^{\mathrm{N}}(t)^{\alpha} (1+\gamma)^{1-\alpha} \ell^{\mathrm{N}}(t)^{-\alpha} \tag{A.36}$$

In addition, we have the labor market equilibrium conditions that state that

$$\ell^{\mathrm{O}}(t) = 1 - h(t) - s(t) \tag{A.37}$$

$$\ell^{\mathrm{S}}(t) = s(t) \tag{A.38}$$

As in the pre–technological revolution, we have that $r = \rho$ and the economy should jump on its steady state level. Let us assume that this is indeed the case, such that the capital labor ratio in the new technology is given by

$$\frac{k^{\mathrm{N}}}{s} = (1+\gamma)\left(\frac{\alpha}{r+\delta}\right)^{\frac{1}{1-\alpha}}$$

and the rental rate by

$$q^{\mathrm{N}} = r + \delta$$

Hence, the wage rate in the new technology is given by

$$w^{\mathrm{N}} = (1-\alpha)(1+\gamma)\left(\frac{\alpha}{r+\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

Then, from (A.29) and (A.30) evaluated at steady state, we get

$$w^{\mathrm{O}} = \frac{\Omega}{r+\Omega} w^{\mathrm{N}}$$

such that

$$\frac{k^{\mathrm{O}}}{1-s-e} = \left(\frac{\Omega(1+\gamma)}{r+\Omega}\right)^{\frac{1}{\alpha}} \left(\frac{\alpha}{r+\delta}\right)^{\frac{1}{1-\alpha}}$$

Then the rental rate in the old technology is given by

$$q^{\mathrm{O}} = \left(\frac{r+\Omega}{\Omega(1+\gamma)}\right)^{\frac{\alpha}{1-\alpha}} (r+\delta)$$

But since, by assumption, $\Omega\gamma/r > 1$, the term in front of $(r+\delta)$ is lower than 1. Therefore, we have

$$q^{\mathrm{O}} < q^{\mathrm{N}}$$

In other words, once the new technology starts to be adopted, it is not worth investing in the old technology. Then $i^{\mathrm{O}}(t) = 0 \ \forall t > t^\star$. Then, equation (A.18) implies that the capital in the old technology evolves as

$$k^{\mathrm{O}}(t) = k^{\mathrm{O}}(t^\star)e^{-(\delta+n)(t-t^\star)}, \ \forall t > t^\star \tag{A.39}$$

On the contrary, it is always worthwhile to invest in the new technology and we therefore have

$$i^{\mathrm{N}}(t) > 0 \text{ and } \lambda_i^{\mathrm{N}}(t) = 0$$

Then (A.25), (A.26) and (A.28) imply that

$$q^{\mathrm{N}}(t) = r + \delta$$

Using (A.35), we then obtain

$$k^{\mathrm{N}}(t) = (1+\gamma)\left(\frac{\alpha}{r+\delta}\right)^{\frac{1}{1-\alpha}} s(t) \tag{A.40}$$

from which we get

$$w^{\mathrm{N}}(t) = (1-\alpha)(1+\gamma)\left(\frac{\alpha}{r+\delta}\right)^{\frac{\alpha}{1-\alpha}} = (1-\alpha)\theta \tag{A.41}$$

where $\theta \equiv (1+\gamma)\left(\frac{\alpha}{r+\delta}\right)^{\frac{\alpha}{1-\alpha}}$.

The labor allocation problem is summarized by equations (A.29) and (A.30), which can be combined to give

$$\dot{\lambda}^s(t) = -\lambda^b(t)w^{\mathrm{N}}(t) + (r+\Omega)\lambda^s(t)$$

Since both $\lambda^b(t)$ and $w^{\mathrm{N}}(t)$ jump to their steady state value, this differential equation is easily solved to give

$$\lambda^s(t) = \frac{\lambda^b(t)w^{\mathrm{N}}(t)}{r+\Omega} \tag{A.42}$$

which can then be used in (A.29) to yield

$$w^{\mathrm{O}}(t) = \frac{\Omega}{r+\Omega}w^{\mathrm{N}}(t) = (1-\alpha)\frac{\Omega\theta}{r+\Omega} \tag{A.43}$$

Then using (A.34) and (A.37), we obtain an expression for $h(t)$

$$h(t) = 1 - s(t) - \left(\frac{r+\Omega}{\Omega\theta}\right)^{\frac{1}{\alpha}} k^{\mathrm{O}}(t)$$

which rewrites

$$h(t) = 1 - s(t) - \left(\frac{r+\Omega}{\Omega\theta}\right)^{\frac{1}{\alpha}} k^{\mathrm{O}}(t^\star)^{-(\delta+n)(t-t^\star)} \tag{A.44}$$

Plugging the latter expression in the law of motion of skilled labor (A.20), we face the following differential equation

$$\dot{s}(t) + \Omega s(t) = \Omega\left(1 - \left(\frac{r+\Omega}{\Omega\theta}\right)^{\frac{1}{\alpha}} k^{\mathrm{O}}(t^\star)^{-(\delta+n)(t-t^\star)}\right)$$

where we made use of the fact that rational expectations imposes that $\overline{s}(t) = s(t)$ in equilibrium. Then, the previous integral rewrites

$$\int_{t^\star}^{t} e^{\Omega(\tau-t^\star)}\left(\dot{s}(\tau)+\Omega s(\tau)\right)\mathrm{d}\tau = \int_{t^\star}^{t} e^{\Omega(\tau-t^\star)}\Omega\left(1-\left(\frac{r+\Omega}{\Omega\theta}\right)^{\frac{1}{\alpha}} k^{\mathrm{O}}(t^\star)e^{-(n+\delta)(\tau-t^\star)}\right)\mathrm{d}\tau$$

we therefore have

$$e^{\Omega(t-t^\star)}(s(t)+\overline{s}) = e^{\Omega(t-t^\star)}-1-\frac{\Omega}{\Omega-\delta-n}\left(\frac{r+\Omega}{\Omega\theta}\right)^{\frac{1}{\alpha}} k^{\mathrm{O}}(t^\star)\left(e^{-(\delta+n-\Omega)(t-t^\star)}-1\right)$$

or

$$s(t) = 1-e^{-\Omega(t-t^\star)}-\frac{\Omega}{\Omega-\delta-n}\left(\frac{r+\Omega}{\Omega\theta}\right)^{\frac{1}{\alpha}} k^{\mathrm{O}}(t^\star)\left(e^{-(\delta+n)(t-t^\star)}-e^{-\Omega(t-t^\star)}\right)$$

where we assume $\overline{s} = 0$. Then, plugging the definition of θ in this expression, we get

$$s(t) = 1-e^{-\Omega(t-t^\star)}-\frac{\Omega}{\Omega-\delta-n}\left(\frac{r+\Omega}{\Omega(1+\gamma)}\right)^{\frac{1}{\alpha}}\left(\frac{r+\delta}{\alpha}\right)^{\frac{1}{1-\alpha}} k^{\mathrm{O}}(t^\star)\left(e^{-(\delta+n)(t-t^\star)}-e^{-\Omega(t-t^\star)}\right)$$

This corresponds to the solution for $s(t)$ reported in the body text.

It is then straightforward to recover the evolution of total output–per–worker, which is given by

$$y(t) = y^{\mathrm{O}}(t)+y^{\mathrm{N}}(t)$$

Output–per–worker in the old technology can be computed by noting that (A.34) implies

$$1-h(t)-s(t) = \left(\frac{1-\alpha}{w^{\mathrm{O}}(t)}\right)^{\frac{1}{\alpha}} k^{\mathrm{O}}(t) = \left(\frac{r+\Omega}{\theta\Omega}\right)^{\frac{1}{\alpha}} k^{\mathrm{O}}(t)$$

Then plugging this result in the production function of the old technology, we have

$$y^{\mathrm{O}}(t) = \left(\frac{r+\Omega}{\theta\Omega}\right)^{\frac{1-\alpha}{\alpha}} k^{\mathrm{O}}(t)$$

or

$$y^{\mathrm{O}}(t) = \left(\frac{r+\Omega}{\theta\Omega}\right)^{\frac{1-\alpha}{\alpha}} k^{\mathrm{O}}(t^\star)e^{-(\delta+n)(t-t^\star)}$$

Likewise, using (A.40) in the production function of the new technology, we have

$$y^{\mathrm{N}}(t) = \theta s(t)$$

Therefore, total output–per–worker is given by

$$y(t) = \theta+\left(\frac{r+\Omega}{\Omega\theta}\right)^{\frac{1}{\alpha}}\theta k(t^\star)\left(\frac{\Omega}{r+\Omega}-\frac{\Omega}{\Omega-\delta-n}\right)e^{-(\delta+n)(t-t^\star)}+\left(\frac{\Omega}{\Omega-\delta-n}\left(\frac{r+\Omega}{\Omega\theta}\right)^{\frac{1}{\alpha}}\theta k(t^\star)-\theta\right)e^{-\Omega(t-t^\star)}$$

Plugging the definition of θ in the last equation, we get

$$\begin{aligned} y(t) =&(1+\gamma)\left(\frac{\alpha}{r+\delta}\right)^{\frac{\alpha}{1-\alpha}} + \left(\frac{r+\Omega}{\Omega(1+\gamma)}\right)^{\frac{1}{\alpha}} \frac{(1+\gamma)(r+\delta)}{\alpha}\left(\frac{\Omega}{r+\Omega} - \frac{\Omega}{\Omega-\delta-n}\right) k^{\mathrm{O}}(t^\star) e^{-(\delta+n)(t-t^\star)} \\ &+ (1+\gamma)\left(\frac{\alpha}{r+\delta}\right)^{\frac{\alpha}{1-\alpha}} \left(\frac{\Omega}{\Omega-\delta-n}\left(\frac{r+\Omega}{\Omega(1+\gamma)}\right)^{\frac{1}{\alpha}}\left(\frac{r+\delta}{\alpha}\right)^{\frac{1}{1-\alpha}} k^{\mathrm{O}}(t^\star) - 1\right) e^{-\Omega(t-t^\star)} \end{aligned}$$

which corresponds to the law of motion of output per worker reported in the main text.

B Proof of proposition

Proof of Proposition 1: Since output per worker is independent from n, the proof is immediate.

Proof of Proposition 2: The steady state output in the economy where only the old technology is available is given by

$$\overline{y}^{\mathrm{O}} = \left(\frac{\alpha}{r+\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

Let us introduce the new technology. The immediate effect of this introduction is that output jumps to the new level

$$y(t^\star) = \left(\frac{r+\Omega}{\Omega(1+\gamma)}\right)^{\frac{1-\alpha}{\alpha}} \left(\frac{\alpha}{r+\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

where we set $k^{\mathrm{O}}(t^\star)$ to the steady state value of the economy with the old technology.

$y(t^\star)$ actually rewrites

$$y(t^\star) = \left(\frac{r+\Omega}{\Omega(1+\gamma)}\right)^{\frac{1-\alpha}{\alpha}} \overline{y}^{\mathrm{O}}$$

Hence, since by assumption $\Omega\gamma/r > 1$, we have

$$y(t^\star) < \overline{y}^{\mathrm{O}}$$

Q.E.D □

Proof of Proposition 3: In order to prove the first part of the proposition, it is convenient to rewrite the dynamics of output–per–worker as

$$y(t) = \psi_0 + \left(\psi_1 - \frac{\psi_2}{\Omega-\delta-n}\right) e^{-(n+\delta)(t-t^\star)} + \left(\frac{\psi_2}{\Omega-\delta-n} - \psi_0\right) e^{-\Omega(t-t^\star)}$$

where

$$\psi_0 =(1+\gamma)\left(\frac{\alpha}{r+\delta}\right)^{\frac{\alpha}{1-\alpha}}$$
$$\psi_1 =\left(\frac{r+\Omega}{\Omega(1+\gamma)}\right)^{\frac{1}{\alpha}}\frac{(1+\gamma)(r+\delta)}{\alpha}\frac{\Omega}{r+\Omega}k^{\mathrm{O}}(t^\star)$$
$$\psi_2 =\left(\frac{r+\Omega}{\Omega(1+\gamma)}\right)^{\frac{1}{\alpha}}\frac{(1+\gamma)(r+\delta)}{\alpha}\Omega k^{\mathrm{O}}(t^\star)$$

What we need to show is that the larger the rate of population growth the lower the growth in output–per–worker at the beginning of the transition, which amounts to prove that the first order derivative of the rate of growth of output per worker with respect to n is negative when t is close to $t^\star$.

The rate of growth of output per worker writes as

$$\frac{\dot{y}(t)}{y(t)} = -\frac{(\delta+n)\left(\psi_1-\frac{\psi_2}{\Omega-\delta-n}\right)e^{-(n+\delta)(t-t^\star)}+\Omega\left(\frac{\psi_2}{\Omega-\delta-n}-\psi_0\right)e^{-\Omega(t-t^\star)}}{\psi_0+\left(\psi_1-\frac{\psi_2}{\Omega-\delta-n}\right)e^{-(n+\delta)t}+\left(\frac{\psi_2}{\Omega-\delta-n}-\psi_0\right)e^{-\Omega(t-t^\star)}} = -\frac{u(t,n)}{v(t,n)}$$

We therefore have that

$$\frac{\partial \dot{y}(t)/y(t)}{\partial n} = -\frac{\frac{\partial u(t,n)}{\partial n}v(t,n)-\frac{\partial n(t,n)}{\partial n}u(t,n)}{v(t,n)^2}$$

Straightforward calculation gives

$$\frac{\partial u(t,n)}{\partial n} =\left(\psi_1-\frac{\psi_2}{\Omega-\delta-n}-\frac{\psi_2(\delta+n)}{(\Omega-\delta-n)^2}-(\delta+n)t\left(\psi_1-\frac{\psi_2}{\Omega-\delta-n}\right)\right)e^{-(n+\delta)(t-t^\star)}+$$
$$+\frac{\Omega\psi_2}{(\Omega-\delta-n)^2}e^{-\Omega(t-t^\star)}$$
$$\frac{\partial v(t,n)}{\partial n} =\left(-\frac{\psi_2}{(\Omega-\delta-n)^2}-t\left(\psi_1-\frac{\psi_2}{\Omega-\delta-n}\right)\right)e^{-(n+\delta)(t-t^\star)}+\frac{\psi_2}{(\Omega-\delta-n)^2}e^{-\Omega(t-t^\star)}$$

At the time of introduction of the new technology, $t=t^\star$, we therefore have $u(t,n)=(\delta+n)\psi_1-\Omega\psi_0+\psi_2$, $v(t,n)=\psi_1$, $\dfrac{\partial u(t,n)}{\partial n}=\psi_1$ and $\dfrac{\partial v(t,n)}{\partial n}=0$. Hence, plugging these results into the derivative of the rate of growth of output per worker evaluated at time $t^\star$, we get

$$\left.\frac{\partial \dot{y}(t)/y(t)}{\partial n}\right|_{t=t^\star} = -1$$

which proves the first part of the proposition.

The second part of the proposition is trivial, as the steady state of the economy does not depend on the rate of population growth. To see this, note that as t goes to infinity, the capital in the old technology, and therefore the quantity of output produced with the old technology, tends to 0. In contrast, the capital stock in the new technology tends to

$$\widetilde{k}^{\mathrm{N}} = (1+\gamma)\left(\frac{\alpha}{r+\delta}\right)^{\frac{1}{1-\alpha}}$$

since s tends to 1. This capital stock, and therefore output in the new technology, is independent from n. So no matter n, all economies will tend to the same limit. Since the rate of growth in the high population growth economy is lower at the beginning of the transition than in the low population growth economy, it has to be greater at some point to converge to the same steady state.

Q.E.D □

C Routinized Technology Model

The household side of the model is very similar to the very first model and des not desserve any particular comments. Among the predictions of the model, it will be that once the new, routinized, technology is implemented, it is not worthwhile anymore to invest in the old technology, such that the law of motion of the old capital will simply be given by (in intensive terms)

$$k_x^{\mathrm{O}}(t) = k_x^{\mathrm{O}}(t^\star)e^{-(\delta+n)(t-t^\star)}$$

Before going into the differences between the non–routinized and the routinized technology, it is useful to setup the invariant parts of the model. First note that the model is that of a small open economy, which implies that at long as it is used by the firm, the capital stock can be financed on international capital markets, such that its rental rate then equates $\rho+\delta$.

Final good sector: The firm in the final good sector maximizes profits so as to determine its demand for each good x and z, at price p_x and p_z.

$$\max x(t)^\varphi z(t)^{1-\varphi} - p_x(t)x(t) - p_z(t)z(t)$$

which leads to the standard demand function

$$p_x(t) = \varphi\left(\frac{x(t)}{z(t)}\right)^{\varphi-1} \tag{C.1}$$

$$p_z(t) = (1-\varphi)\left(\frac{x(t)}{z(t)}\right)^{\varphi} \tag{C.2}$$

C.1 The Model without the Routinized Technology

As implicit in the very beginning of the section, we will not present the household problem which is very similar to that in the first model, and will rather focus on the firms problem. All variables will be expressed in intensive terms.

Intermediate good z: The firm determines its production plans maximizing its profit function

$$p_z(t)A_zk_z(t)^{\alpha}(1-\ell_x^{\mathrm{O}}(t))^{1-\alpha}-q_z(t)k_z(t)-w_z(t)(1-\ell_x^{\mathrm{O}}(t))$$

which leads to the following demand functions.

$$\alpha p_z(t)A_zk_z(t)^{\alpha-1}(1-\ell_x^{\mathrm{O}}(t))^{1-\alpha} = q_z(t) \tag{C.3}$$

$$(1-\alpha)p_z(t)A_zk_z(t)^{\alpha}(1-\ell_x^{\mathrm{O}}(t))^{-\alpha} = w_z(t) \tag{C.4}$$

Intermediate good x: The firm determines its production plans maximizing its profit function

$$p_x(t)k_x^{\mathrm{O}}(t)^{\alpha}\left(e\left(\frac{w_x^{\mathrm{O}}(t)}{w_z(t)}\right)\ell_x^{\mathrm{O}}(t)\right)^{1-\alpha}-q_x^{\mathrm{O}}(t)k_x^{\mathrm{O}}(t)-w_x^{\mathrm{O}}(t)\ell_x^{\mathrm{O}}(t)$$

which leads to the following demand functions.

$$\alpha p_x(t)k_x^{\mathrm{O}}(t)^{\alpha-1}\left(e\left(\frac{w_x^{\mathrm{O}}(t)}{w_z(t)}\right)\ell_x^{\mathrm{O}}(t)\right)^{1-\alpha} = q_x^{\mathrm{O}}(t) \tag{C.5}$$

$$(1-\alpha)p_x(t)k_x^{\mathrm{O}}(t)^{\alpha}e\left(\frac{w_x^{\mathrm{O}}(t)}{w_z(t)}\right)^{1-\alpha}\ell_x^{\mathrm{O}}(t)^{-\alpha} = w_x^{\mathrm{O}}(t) \tag{C.6}$$

$$(1-\alpha)p_x(t)k_x^{\mathrm{O}}(t)^{\alpha}\frac{1}{w_z(t)}e'\left(\frac{w_x^{\mathrm{O}}(t)}{w_z(t)}\right)e\left(\frac{w_x^{\mathrm{O}}(t)}{w_z(t)}\right)^{-\alpha}\ell_x^{\mathrm{O}}(t)^{1-\alpha} = \ell_x^{\mathrm{O}}(t) \tag{C.7}$$

From (C.6) and (C.7), we recover the standard Solow condition

$$\frac{w_x^{\mathrm{O}}(t)}{w_z(t)}\frac{e'\left(\frac{w_x^{\mathrm{O}}(t)}{w_z(t)}\right)}{e\left(\frac{w_x^{\mathrm{O}}(t)}{w_z(t)}\right)}=1$$

which indicates that in equilibrium, we will have

$$w_x^{\mathrm{O}}(t)=\gamma^{\star}w_z(t) \text{ with } \gamma^{\star}>1 \tag{C.8}$$

and that determines an optimal effort level, $e^{\star}=e(\gamma^{\star})$.

Determination of output–per–worker From (C.3), (C.6), and (C.8), we have

$$\ell_x^{\mathrm{O}}(t)=\frac{\varphi}{\varphi+(1-\varphi)\gamma^{\star}}$$

Then using the fact that both capital are in use, and that their rental rates are both equalized to $\rho+\delta$, we have from (C.4) and (C.5), and making use of (C.1) and (C.2):

$$k_z(t)=\frac{1-\varphi}{\varphi}k_x^{\mathrm{O}}(t) \tag{C.9}$$

Let us then compute the ratio $x(t)/z(t)$:

$$\frac{x(t)}{z(t)} = \frac{1}{A_z}\left(\frac{k_x^{\mathrm{O}}(t)}{k_z(t)}\right)^{\alpha}\left(\frac{e^\star \ell_x^{\mathrm{O}}(t)}{1-\ell_x^{\mathrm{O}}(t)}\right)^{1-\alpha} = \frac{\varphi}{A_z(1-\varphi)}\left(\frac{e^\star}{\gamma^\star}\right)^{1-\alpha}$$

which then implies that

$$p_x(t) = \varphi^{\varphi}(1-\varphi)^{1-\varphi}A_z^{1-\varphi}\left(\frac{\gamma^\star}{e^\star}\right)^{(1-\alpha)(1-\varphi)}$$
$$p_z(t) = \varphi^{\varphi}(1-\varphi)^{1-\varphi}A_z^{1-\varphi}\left(\frac{e^\star}{\gamma^\star}\right)^{(1-\alpha)\varphi}$$

Then using (C.4) evaluated at $q_z(t) = r+\delta$, and plugging the definition of $\ell_x^{\mathrm{O}}(t)$, w obtain

$$k_z(t) = (1-\varphi)\left(\frac{\alpha}{r+\delta}\right)^{\frac{1}{1-\alpha}}\left(\varphi^{\varphi}(1-\varphi)^{1-\varphi}A_z^{1-\varphi}\right)^{\frac{1}{1-\alpha}}\frac{e^{\star\varphi}\gamma^{\star 1-\varphi}}{\varphi+(1-\varphi)\gamma^\star}$$

Then, using (C.9), we obtain

$$k_x^{\mathrm{O}}(t) = \varphi\left(\frac{\alpha}{r+\delta}\right)^{\frac{1}{1-\alpha}}\left(\varphi^{\varphi}(1-\varphi)^{1-\varphi}A_z^{1-\varphi}\right)^{\frac{1}{1-\alpha}}\frac{e^{\star\varphi}\gamma^{\star 1-\varphi}}{\varphi+(1-\varphi)\gamma^\star}$$

Using these results in the definition of the production function for x and z and aggregating to form $y(t) = x(t)^{\varphi}z(t)^{1-\varphi}$, we get

$$y(t) = \left(\frac{\alpha}{r+\delta}\right)^{\frac{\alpha}{1-\alpha}}\left(\varphi^{\varphi}(1-\varphi)^{1-\varphi}A_z^{1-\varphi}\right)^{\frac{\alpha}{1-\alpha}}\frac{e^{\star\varphi}\gamma^{\star 1-\varphi}}{\varphi+(1-\varphi)\gamma^\star}$$

C.2 The Model with the Routinized Technology

Intermediate good z: The firm determines its production plans maximizing its profit function

$$p_z(t)A_z k_z(t)^{\alpha}(1-\ell_x^{\mathrm{O}}(t)-\ell_x^{\mathrm{N}}(t))^{1-\alpha} - q_z(t)k_z(t) - w_z(t)(1-\ell_x^{\mathrm{O}}(t)-\ell_x^{\mathrm{N}}(t))$$

which leads to the following demand functions.

$$\alpha p_z(t)A_z k_z(t)^{\alpha-1}(1-\ell_x^{\mathrm{O}}(t)-\ell_x^{\mathrm{N}}(t))^{1-\alpha} = q_z(t) \tag{C.10}$$
$$(1-\alpha)p_z(t)A_z k_z(t)^{\alpha}(1-\ell_x^{\mathrm{O}}(t)-\ell_x^{\mathrm{N}}(t))^{-\alpha} = w_z(t) \tag{C.11}$$

Intermediate good x: The firm determines its production plans maximizing its profit function

$$p_x(t)\left(k_x^{\mathrm{O}}(t)^{\alpha}\left(e\left(\frac{w_x^{\mathrm{O}}(t)}{w_z(t)}\right)\ell_x^{\mathrm{O}}(t)\right)^{1-\alpha} + k_x^{\mathrm{N}}(t)^{\alpha}\left(\widetilde{e}\ell_x^{\mathrm{N}}(t)\right)^{1-\alpha}\right) - q_x^{\mathrm{O}}(t)k_x^{\mathrm{O}}(t) - w_x^{\mathrm{O}}(t)\ell_x^{\mathrm{O}}(t) - q_x^{\mathrm{N}}(t)k_x^{\mathrm{N}}(t) - w_x^{\mathrm{N}}(t)\ell_x^{\mathrm{N}}(t)$$

which leads to the following demand functions.

$$\alpha p_x(t)k_x^{\mathrm{O}}(t)^{\alpha-1}\left(e\left(\frac{w_x^{\mathrm{O}}(t)}{w_z(t)}\right)\ell_x^{\mathrm{O}}(t)\right)^{1-\alpha} = q_x^{\mathrm{O}}(t) \tag{C.12}$$

$$\alpha p_x(t)k_x^{\mathrm{N}}(t)^{\alpha-1}\left(\widetilde{e}\ell_x^{\mathrm{N}}(t)\right)^{1-\alpha} = q_x^{\mathrm{N}}(t) \tag{C.13}$$

$$(1-\alpha)p_x(t)k_x^{\mathrm{O}}(t)^{\alpha}e\left(\frac{w_x^{\mathrm{O}}(t)}{w_z(t)}\right)^{1-\alpha}\ell_x^{\mathrm{O}}(t)^{-\alpha} = w_x^{\mathrm{O}}(t) \tag{C.14}$$

$$(1-\alpha)p_x(t)k_x^{\mathrm{O}}(t)^{\alpha}\frac{1}{w_z(t)}e'\left(\frac{w_x^{\mathrm{O}}(t)}{w_z(t)}\right)e\left(\frac{w_x^{\mathrm{O}}(t)}{w_z(t)}\right)^{-\alpha}\ell_x^{\mathrm{O}}(t)^{1-\alpha} = \ell_x^{\mathrm{O}}(t) \tag{C.15}$$

$$(1-\alpha)p_x(t)k_x^{\mathrm{N}}(t)^{\alpha}\widetilde{e}^{1-\alpha}\ell_x^{\mathrm{N}}(t)^{-\alpha} = w_x^{\mathrm{N}}(t) \tag{C.16}$$

$$\tag{C.17}$$

From (C.14) and (C.15), we recover the standard Solow condition

$$\frac{w_x^{\mathrm{O}}(t)}{w_z(t)}\frac{e'\left(\frac{w_x^{\mathrm{O}}(t)}{w_z(t)}\right)}{e\left(\frac{w_x^{\mathrm{O}}(t)}{w_z(t)}\right)} = 1$$

which indicates that in equilibrium, we will have

$$w_x^{\mathrm{O}}(t) = \gamma^\star w_z(t) \text{ with } \gamma^\star > 1 \tag{C.18}$$

and that determines an optimal effort level, $e^\star = e(\gamma^\star)$.

Determination of output–per–worker First of all note that since both the z and the x sector endowed with the new technology are perfectly competitive, we have

$$w_x^{\mathrm{N}}(t) = w_z(t) \tag{C.19}$$

$$q_x^{\mathrm{N}}(t) = q_z(t) \tag{C.20}$$

These two equations, (C.10) , (C.11), (C.13) and (C.16) trigger that

$$\frac{k_x^{\mathrm{N}}(t)}{\ell_x^{\mathrm{N}}(t)} = \frac{k_z(t)}{1-\ell_x^{\mathrm{O}}(t)-\ell_x^{\mathrm{N}}(t)} \tag{C.21}$$

Furthermore, from (C.14), (C.16) and (C.19)

$$\frac{k_x^{\mathrm{O}}(t)}{\ell_x^{\mathrm{O}}(t)} = \left(\frac{\gamma^\star\widetilde{e}}{e^\star}\right)^{\frac{1}{\alpha}}\frac{e^\star}{\widetilde{e}}\frac{k_x^{\mathrm{N}}(t)}{\ell_x^{\mathrm{N}}(t)} \tag{C.22}$$

Using (C.21) in (C.19), we obtain

$$p_x(t) = \frac{A_z}{\widetilde{e}^{1-\alpha}}p_z(t)$$

Then using (C.1) and (C.2), we have

$$\frac{p_x(t)}{p_z(t)} = \frac{\varphi}{1-\varphi}\frac{z(t)}{x(t)} = \frac{A_z}{\widetilde{e}^{1-\alpha}} \iff \frac{x(t)}{z(t)} = \frac{\varphi}{1-\varphi}\frac{\widetilde{e}^{1-\alpha}}{A_z} \tag{C.23}$$

from which we get

$$p_x(t) = \varphi^{\varphi}(1-\varphi)^{1-\varphi}\left(\frac{A_z}{\widetilde{e}^{1-\alpha}}\right)^{1-\varphi} \tag{C.24}$$

$$p_z(t) = \varphi^{\varphi}(1-\varphi)^{1-\varphi}\left(\frac{\widetilde{e}^{1-\alpha}}{A_z}\right)^{\varphi} \tag{C.25}$$

Using the fact that as soon as the new technology is implemented, it is not while to invest in it, we have

$$q_x^{\mathrm{N}}(t) = r + \delta$$

Then using (C.13), we obtain

$$k_x^{\mathrm{N}}(t) = p_x(t)^{\frac{1}{1-\alpha}}\left(\frac{\alpha}{r+\delta}\right)^{\frac{1}{1-\alpha}}\widetilde{e}\ell_x^{\mathrm{N}}(t) \tag{C.26}$$

Then using (C.22) and (C.21), we get

$$k_x^{\mathrm{O}}(t) = \left(\frac{\gamma^{\star}\widetilde{e}}{e^{\star}}\right)^{\frac{1}{\alpha}} e^{\star}p_x(t)^{\frac{1}{1-\alpha}}\left(\frac{\alpha}{r+\delta}\right)^{\frac{1}{1-\alpha}}\ell_x^{\mathrm{O}}(t) \tag{C.27}$$

$$k_z(t) = A_z p_x(t)^{\frac{1}{1-\alpha}}\left(\frac{\alpha}{r+\delta}\right)^{\frac{1}{1-\alpha}}\widetilde{e}^{\alpha}(1-\ell_x^{\mathrm{N}}(t)-\ell_x^{\mathrm{O}}(t)) \tag{C.28}$$

Then using the production function in the new technology we get

$$x^{\mathrm{N}}(t) = p_x(t)^{\frac{\alpha}{1-\alpha}}\left(\frac{\alpha}{r+\delta}\right)^{\frac{\alpha}{1-\alpha}}\widetilde{e}\ell_x^{\mathrm{N}}(t) \tag{C.29}$$

$$x^{\mathrm{O}}(t) = p_x(t)^{\frac{\alpha}{1-\alpha}}\left(\frac{\alpha}{r+\delta}\right)^{\frac{\alpha}{1-\alpha}}\gamma^{\star}\widetilde{e}\ell_x^{\mathrm{O}}(t) \tag{C.30}$$

$$z(t) = A_z p_x(t)^{\frac{\alpha}{1-\alpha}}\left(\frac{\alpha}{r+\delta}\right)^{\frac{\alpha}{1-\alpha}}\widetilde{e}^{\alpha}(1-\ell_x^{\mathrm{N}}(t)-\ell_x^{\mathrm{O}}(t)) \tag{C.31}$$

Then, using these results together with

$$x(t) = x^{\mathrm{O}}(t) + x^{\mathrm{N}}(t) = \frac{\varphi}{1-\varphi}\frac{\widetilde{e}^{1-\alpha}}{A_z}z(t)$$

we find

$$\ell_x^{\mathrm{N}}(t) = \varphi - (\varphi + \gamma^{\star}(1-\varphi))\ell_x^{\mathrm{O}}(t) \tag{C.32}$$

Therefore, plugging this result in (C.30) and (C.29), one gets

$$x(t) = \varphi p_x(t)^{\frac{\alpha}{1-\alpha}}\left(\frac{\alpha}{r+\delta}\right)^{\frac{\alpha}{1-\alpha}}\widetilde{e}(1+(\gamma^{\star}-1)\ell_x^{\mathrm{O}}(t))$$

Then since $x(t) = \frac{\varphi}{1-\varphi}\frac{\widetilde{e}^{1-\alpha}}{A_z}z(t)$, we have

$$y(t) = \left(\frac{1-\varphi}{\varphi}\frac{A_z}{\widetilde{e}^{1-\alpha}}\right)^{1-\varphi}x(t)$$

such that

$$y(t)=\left(\varphi^{\varphi}(1-\varphi)^{1-\varphi}\left(\frac{A_z}{\widetilde{e}^{1-\alpha}}\right)^{1-\varphi}\right)p_x(t)^{\frac{\alpha}{1-\alpha}}\left(\frac{\alpha}{r+\delta}\right)^{\frac{\alpha}{1-\alpha}}\widetilde{e}(1+(\gamma^\star-1)\ell_x^{\mathrm{O}}(t))$$

We still have to determine the evolution of $\ell_x^{\mathrm{O}}(t)$,which can be obtained from $w_x^{\mathrm{O}}(t)=\gamma^\star w_x^{\mathrm{N}}(t)$, which implies that

$$\ell_x^{\mathrm{O}}(t)=\left(\frac{r+\delta}{\alpha}\right)^{\frac{1}{1-\alpha}}\left(\frac{e^\star}{\gamma^\star\widetilde{e}}\right)^{\frac{1}{\alpha}}\frac{\widetilde{e}}{e^\star}\frac{1}{\widetilde{e}^{\varphi}}\frac{k_x^{\mathrm{O}}(t)}{\left(\varphi^{\varphi}(1-\varphi)^{1-\varphi}A_z^{1-\varphi}\right)^{\frac{1}{1-\alpha}}}$$

Then plugging this result in the definition of $y(t)$ and using the definition of $p_x(t)$ (equation (C.24)) and the law of motion of $k_x^{\mathrm{O}}(t)$, we obtain output per worker as a function of time:

$$y(t)=\left(\varphi^{\varphi}(1-\varphi)^{1-\varphi}A_z^{1-\varphi}\right)^{\frac{1}{1-\alpha}}\left(\frac{\alpha}{r+\delta}\right)^{\frac{\alpha}{1-\alpha}}\widetilde{e}^{\varphi}+\frac{\gamma^\star-1}{\gamma^\star}\frac{r+\delta}{\alpha}\left(\frac{e^\star}{\gamma^\star\widetilde{e}}\right)^{\frac{1-\alpha}{\alpha}}k_x^{\mathrm{O}}(t^\star)e^{-(\delta+n)(t-t^\star)}$$

Is it worth implementing the routinized technology? The answer to this question rests on comparing the wage rate paid to workers in sector X with the old technology before and after the introduction of the routinized technology (before and after $t=t^\star$).

$$\begin{aligned}
w_x^{\mathrm{O}}(t)=w_1&\equiv(1-\alpha)(\varphi^{\varphi}(1-\varphi)^{1-\varphi}A_z^{1-\varphi})^{\frac{1}{1-\alpha}}\left(\tfrac{\alpha}{r+\delta}\right)^{\frac{\alpha}{1-\alpha}}\gamma^{\star 1-\varphi}e^{\star\varphi} && \text{for } t<t^\star\\
w_x^{\mathrm{O}}(t)=w_2&\equiv(1-\alpha)(\varphi^{\varphi}(1-\varphi)^{1-\varphi}A_z^{1-\varphi})^{\frac{1}{1-\alpha}}\left(\tfrac{\alpha}{r+\delta}\right)^{\frac{\alpha}{1-\alpha}}\left(\tfrac{\gamma^\star}{e^\star}\right)^{1-\varphi}\gamma^\star\widetilde{e} && \text{for } t\geqslant t^\star
\end{aligned}$$

It is then worth implementing the routinized technology if $w_2\geqslant w_1$, or

$$\frac{\gamma^\star\widetilde{e}}{e^\star}\geqslant 1$$

which is the assumption we placed on $\widetilde{e}$.

Labor productivity growth: Note that, after the introduction of the routinized technology, output–per–worker can be rewritten as

$$y(t)=\psi_0+\psi_1 e^{-(\delta+n)(t-t^\star)}$$

with $\psi_0=\left(\varphi^{\varphi}(1-\varphi)^{1-\varphi}A_z^{1-\alpha}\right)^{\frac{1}{1-\alpha}}\left(\frac{\alpha}{r+\delta}\right)^{\frac{\alpha}{1-\alpha}}\widetilde{e}^{\varphi}>0$ and $\psi_1=\frac{\gamma^\star-1}{\gamma^\star}\frac{r+\delta}{\alpha}\left(\frac{e^\star}{\gamma^\star\widetilde{e}}\right)^{\frac{1-\alpha}{\alpha}}k_x^{\mathrm{O}}(t^\star)>0$. The labor productivity growth is given by

$$\frac{\dot{y}(t)}{y(t)}=-\frac{\psi_1(\delta+n)e^{-(\delta+n)(t-t^\star)}}{\psi_0+\psi_1 e^{-(\delta+n)(t-t^\star)}}=-\frac{u(t,n)}{v(t,n)}$$

We therefore have that

$$\frac{\partial\dot{y}(t)/y(t)}{\partial n}=-\frac{\frac{\partial u(t,n)}{\partial n}v(t,n)-\frac{\partial n(t,n)}{\partial n}u(t,n)}{v(t,n)^2}$$

Straightforward calculation gives

$$\begin{aligned}\frac{\partial u(t,n)}{\partial n} &= \psi_1 e^{-(\delta+n)(t-t^\star)} - \psi_1(\delta+n)(t-t^\star)e^{-(\delta+n)(t-t^\star)} \\ \frac{\partial v(t,n)}{\partial n} &= -\psi_1(t-t^\star)e^{-(\delta+n)(t-t^\star)}\end{aligned}$$

At the time of introduction of the new technology, $t = t^\star$, we therefore have $u(t,n) = \psi_1(\delta+n)$, $v(t,n) = \psi_0 + \psi_1$, $\frac{\partial u(t,n)}{\partial n} = \psi_1(\delta+n)$ and $\frac{\partial v(t,n)}{\partial n} = 0$. Hence, plugging these results into the derivative of the rate of growth of output per worker evaluated at time $t^\star$, we get

$$\left.\frac{\partial \dot{y}(t)/y(t)}{\partial n}\right|_{t=t^\star} = -\frac{\psi_1}{\psi_0+\psi_1} < 0$$

Hence, at the time of introduction of the routinized technology, the economy with the largest rate of population growth experiences lower growth in labor productivity.

As for the previous model, the steady state of the economy does not depend on the rate of population growth. Since the rate of growth in the high population growth economy is lower at the beginning of the transition than in the low population growth economy, it has to be greater at some point to catch up and converge to the same steady state.